COMMUTATORS AND THE STRONG RADICAL

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1. Introduction. Throughout this note $\mathfrak{H}$ will denote a fixed *separable*, infinite-dimensional, Hilbert space. A von Neumann algebra acting on $\mathfrak{H}$ is a weakly closed, self-adjoint algebra of operators that contains the identity operator I on $\mathfrak{H}$. *Henceforth, it will always be assumed, without further notice, that the von Neumann algebras under consideration act on* $\mathfrak{H}$. An operator A in a von Neumann algebra $\mathfrak{A}$ is a *commutator in* $\mathfrak{A}$ if there exist operators B and C in $\mathfrak{A}$ such that $A = BC - CB$. The problem of specifying which operators in $\mathfrak{A}$ are commutators in $\mathfrak{A}$ has been solved in certain special cases. If $\mathfrak{A}$ is an algebra of type I_n, then the commutators in $\mathfrak{A}$ are exactly the operators with central trace zero [4]. If $\mathfrak{A}$ is a factor of type I_∞, then the non-commutators in $\mathfrak{A}$ are exactly the operators that are congruent to a non-zero scalar modulo the ideal of compact operators [2]. Finally, if $\mathfrak{A}$ is a factor of type III, then the non-commutators in $\mathfrak{A}$ are exactly the non-zero scalar operators [3].

These results form the beginnings of a structure theory for commutators in a general von Neumann algebra. In this note, we continue to develop this structure theory by studying commutators in an arbitrary properly infinite von Neumann algebra $\mathfrak{A}$. Our main theorem is that every operator in the strong radical of $\mathfrak{A}$ is a commutator in $\mathfrak{A}$. On the way to proving this theorem, we obtain some new characterizations of the strong radical.

2. Preliminaries. A von Neumann algebra is *properly infinite* if it contains no non-zero finite central projection. Any properly infinite von Neumann algebra $\mathfrak{A}$ splits naturally into three disjoint subsets as follows. Let $\mathfrak{R}$ denote the *strong radical* of $\mathfrak{A}$, i.e., let $\mathfrak{R}$ be the intersection of all (proper, two-sided) maximal ideals in $\mathfrak{A}$. Let (S) denote the class of all operators A in $\mathfrak{A}$ such that for some maximal ideal $\mathfrak{M}$ of $\mathfrak{A}$, A is congruent to a non-zero scalar modulo $\mathfrak{M}$. Finally, let (F) denote the class of all operators A in $\mathfrak{A}$ such that $A \notin \mathfrak{R} \cup (S)$. It is clear that $\mathfrak{A}$ is the disjoint union $\mathfrak{A} = \mathfrak{R} \cup (S) \cup (F)$; in view of our previous experience with commutators, we make the following conjecture.

Conjecture. The commutators in a properly infinite von Newmann algebra $\mathfrak{A}$ are exactly the operators in the set $\mathfrak{R} \cup (F)$.

The proof of one part of this conjecture is quite easy.

Proposition 2.1. *No operator A in the set (S) is a commutator in $\mathfrak{A}$.*

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