A NOTE ON THE ROGERS—RAMANUJAN IDENTITIES

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The identities in question are ([3, Ch. 6], [4, Ch. 19])

$$\sum_{0}^{\infty} \frac{x^{n^2}}{(x)_n} = \prod_{0}^{\infty} (1 - x^{5n+1})^{-1}(1 - x^{5n+4})^{-1} \tag{1}$$

and

$$\sum_{0}^{\infty} \frac{x^{n(n+1)}}{(x)_n} = \prod_{0}^{\infty} (1 - x^{5n+2})^{-1}(1 - x^{5n+3})^{-1}, \tag{2}$$

where

$$(x)_n = (1 - x)(1 - x^2) \cdots (1 - x^n), \quad (x)_0 = 1.$$

In view of the Jacobi theta formula

$$\sum_{-\infty}^{\infty} x^{n^2} z = \prod_{1}^{\infty} (1 - x^{2n})(1 + x^{2n-1}z)(1 + x^{2n-1}z^{-1}), \tag{3}$$

it is easily verified that (1) and (2) are equivalent to

$$\sum_{0}^{\infty} \frac{x^{n^2}}{(x)_n} = \sum_{-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(5n+1)} \prod_{1}^{\infty} (1 - x^m)^{-1} \tag{4}$$

and

$$\sum_{0}^{\infty} \frac{x^{n(n+1)}}{(x)_n} = \sum_{-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(5n+3)} \prod_{1}^{\infty} (1 - x^m)^{-1}, \tag{5}$$

respectively.

The object of the present note is to give a simplified proof of (4) and (5) which depends only on the identity (3). Essentially this is the proof given by the writer in [1]; however since the discussion in that paper is obscured by the occurrence of other material, it has seemed worthwhile giving a brief but connected account of the proof.

We define the function $I_n(z) = I_n(z, x)$ by means of

$$\prod_{0}^{\infty} (1 + x^r yz)(1 + x^r y^{-1} z) = \sum_{-\infty}^{\infty} y^n I_n(z). \tag{6}$$

The function $I_n(z)$ is a basic analog of the Bessel function first defined by F. H. Jackson [5], [6] and discussed in a recent paper by Hahn [2]. However no properties of $I_n(z)$ will be assumed in the present paper.

It is evident from (6) that

$$I_{-n}(z) = I_n(z). \tag{7}$$

Received June 28, 1967. Supported in part by NSF grant GP-5174.