ON ZERO TYPE SETS OF LAGUERRE POLYNOMIALS

By J. W. Brown

1. Statement of problem. Along with the set $\{L_n^{(a)}(x)\}$ of generalized Laguerre polynomials, consider the modification $\{L_n^{(a+mn)}(x)\}$ where m is an integer. It is known that this modification is of type zero for certain values of m. Recall [3] that zero type polynomial sets $\{\pi_n(x)\}$ may be characterized as those having generating relations of the form

$$\sum_{n=0}^{\infty} \pi_n(x)t^n = \alpha(t) \exp [x\beta(t)] \tag{1}$$

where it is understood that $\alpha(t)$ and $t^{-1}\beta(t)$ have at least formal power series expansions with non-zero initial coefficients. The well-known [2] generating relations

$$\sum_{n=0}^{\infty} L_n^{(a)}(x)t^n = (1 - t)^{-1-a} \exp \left[\frac{-xt}{1 - t}\right] \tag{2}$$

and

$$\sum_{n=0}^{\infty} L_n^{(a-n)}(x)t^n = (1 + t)^a \exp [-xt] \tag{3}$$

as well as the less familiar [4]

$$\sum_{n=0}^{\infty} L_n^{(a+n)}(x)t^n = \frac{[1 + (1 - 4t)^{1/2}]^{-a}}{2^{-a}(1 - 4t)^{1/2}} \exp \left[-x \frac{1 - (1 - 4t)^{1/2}}{1 + (1 - 4t)^{1/2}}\right] \tag{4}$$

and

$$\sum_{n=0}^{\infty} L_n^{(a-2n)}(x)t^n = \frac{[1 + (1 + 4t)^{1/2}]^{1+a}}{2^{1+a}(1 + 4t)^{1/2}} \exp \left[\frac{-2xt}{1 + (1 + 4t)^{1/2}}\right] \tag{5}$$

are evidently of the form (1). So $\{L_n^{(a+mn)}(x)\}$ is of type zero for $-2 \leq m \leq 1$. The purpose of this paper is to show that the restriction on the range of m may be dropped:

Theorem. $\{L_n^{(a+mn)}(x)\}$ *is of type zero for any integer* m.

2. Proof of theorem. Starting with [2]

$$L_n^{(a)}(x) = \frac{(1 + a)_n}{n!} \, {}_1F_1(-n; 1 + a; x)$$

Received June 26, 1967; in revised form July 28, 1967. The author is indebted to the late Professor E. D. Rainville and the University of Michigan for encouraging and supporting this research.