OPERATIONAL EQUATIONS FOR A CLASS OF SYMMETRIC q-POLYNOMIALS

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1. Introduction. The polynomials $H_n(x)$ and $G_n(x)$ defined by

$$\prod_{n=0}^{\infty}(1-q^n u)^{-1}(1-q^n xu)^{-1} = \sum_{n=0}^{\infty} H_n(x)\frac{u^n}{(q)_n}, \tag{1.1}$$

$$\prod_{n=0}^{\infty}(1-q^n u)(1-q^n xu) = \sum_{n=0}^{\infty}(-1)^n G_n(x) q^{n(n-1)/2}\frac{u^n}{(q)_n}, \tag{1.2}$$

where

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n), \qquad (q)_0 = 1,$$

and $|q| < 1$, are obviously polynomials in x with coefficients which are polynomials in q. Properties of these polynomials have been discussed in papers by Bateman [2], Carlitz [4] and [6], Hahn [7], Szegö [9], and Wigert [11].

A generalization of these polynomials is furnished by [1], [10]

$$\prod_{n=0}^{\infty}\prod_{j=1}^{k}(1-q^n x_j u)^{-1} = \sum_{n=0}^{\infty} H_n(x_1, x_2, \cdots, x_k)\frac{u^n}{(q)_n}, \tag{1.3}$$

$$\prod_{n=0}^{\infty}\prod_{j=1}^{k}(1-q^n x_j u) = \sum_{n=0}^{\infty}(-1)^n q^{n(n-1)/2} G_n(x_1, x_2, \cdots, x_k)\frac{u^n}{(q)_n}. \tag{1.4}$$

Clearly the functions H_n and G_n are symmetric polynomials in the variables x_j whose coefficients are polynomials in q. Hence the use of the term symmetric q-polynomials to denote such functions.

For the three variable case, Carlitz [3] modified (1.3) and (1.4) and introduced two additional symmetric q-polynomials. This slight modification furnished polynomials which are symmetric in the three variables and whose coefficients are rational functions of q. He also introduced an operator Ω_r, which suggested operational equations for the symmetric q-polynomials. Two bases for the set of homogeneous symmetric polynomials of weight n were constructed. This result was then used to solve several operational equations and to derive formulas for products of the form $G_m G_n$.

The purpose of the present paper is to obtain similar results for symmetric q-polynomials in four variables. In particular, we define six symmetric q-polynomials and an operator Ω_r, which is a slight modification of the operator Ω_r used by Carlitz. In addition, two new operators, ω_r and ϕ_r, are defined in terms of Ω_r. Operational equations involving Ω_r, ω_r, and ϕ_r are obtained and used to prove that two given sets form bases for homogeneous symmetric polynomials of weight n.

Received June 15, 1967. Supported by National Science Foundation grant GP-5174.