AN EXTENSION OF HYPERBOLIC GEOMETRY AND JULIA'S THEOREM

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Introduction. Let X be a real pre-Hilbert space—inner-product space without assumption of completeness. In this paper we show how certain aspects of hyperbolic geometry in the unit ball of n-dimensional euclidean space carry over to the unit ball of X. In particular it will be seen that Julia's theorem [2; 27] is essentially a theorem about contractions in the hyperbolic metric. In the last section we obtain an inequality for such maps in the unit disk.

1. Let (x, y) be the inner product in X, $|x|$ the norm, and $B(x, R)$, $S(x, R)$, respectively, the ball, sphere, with center x and radius R. $B(0, 1)$, the unit ball, is denoted simply B and the unit sphere, $S(0, 1)$, S. Inversion and reflection can be carried out in X with the same definitions as in finite dimensional euclidean space. The inverse of y with respect to S is $y^* = y/|y|^2$. If $|y| > 1$, $S(y, \sqrt{|y|^2-1})$ is orthogonal to S and inversion with respect to $S(y, \sqrt{|y|^2-1})$ is given by $x \to A_y(x) = y + (|y|^2 - 1)(x - y)/|x - y|^2$. This and a number of other useful formulas about hyperbolic geometry are summarized in [2]. A_y is of order 2, preserves B and S, and interchanges 0 and y^*. Let G be the group generated by inversions in spheres orthogonal to S and reflections in hyperplanes through the origin. B and S are preserved by G. Furthermore, B carries a G-invariant metric $\rho(x, y)$ defined by $\tanh \rho(x, y) = r(x, y)$ where $r(x, y) = |A_{y^*}(x)|$. To see that ρ is actually a metric and G invariant we note that $T \in G$ is 'finitely determined'. That is, T is a product of inversions in spheres with centers say $y_1, \cdots, y_k$ and reflections in the orthogonal complements of $z_1, \cdots, z_m$. Hence if Y is any finite dimensional subspace of X containing $y_1, \cdots, y_k, z_1, \cdots, z_m$, then T preserves $B' = B \cap Y$ and T restricted to B' is a hyperbolic motion. But then to prove G-invariance and metric properties of ρ, one can always restrict to a finite dimensional subspace, where these are known.

Observe that $r(x, y)$ is also a G-invariant metric—this follows from the addition formula for tanh—and is an increasing function of ρ. In practice it is easier to work with r than ρ. By direct calculation one obtains:

$$r(x, y) = \frac{|x - y|}{|y|\,|x - y^*|}, \qquad r(0, y) = |y|, \tag{1}$$

$$\frac{1 - r^2(x, y)}{1 - |y|^2} = \frac{1 - |x|^2}{|y|^2\,|x - y^*|^2} \tag{2}$$

$$r(-|x|, |y|) \leq r(x, y) \leq r(|x|, |y|). \tag{3}$$

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