OPERATORS ON ALGEBRAS OF ARITHMETIC FUNCTIONS

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1. Introduction. We shall use the term *arithmetic function* to refer to any mapping f from the positive integers into the real or complex numbers; $f(n)$ need not express any "arithmetic" property of the integer n. Often in this context the "arithmetic" element is introduced through use of a binary product operation, defined so that the manner in which the functions are combined to form the product depends on divisibility properties of the argument n. Interest then centers on the structure of algebraic systems employing such operations, or on the preservation of "arithmetic" properties (such as multiplicativity) under such operations. In this paper we shall consider two well-known products of this type: the Dirichlet product $f * g(n) = \sum f(d)g(n/d)$, in which the sum is extended over all divisors d of n, and the unitary product $f \times g(n) = \sum' f(d)g(n/d)$, in which $\sum'$ indicates that the sum is extended over only those divisors such that $(d, n/d) = 1$.

In the case of the Dirichlet product it is well known [1], [2], [5] that if we also define, in the usual way, addition and equality of functions, and multiplication by a constant, the set of all arithmetic functions becomes a commutative algebra. In this paper we shall restrict attention primarily to the *real* algebra obtained in this way from real-valued arithmetic functions and real scalars. We let A stand for the set of all real-valued arithmetic functions. P will denote the set of all real functions f such that $f(1) > 0$, while M will denote the set of all real f which are multiplicative, i.e. $f(1) = 1$ and $f(mn) = f(m)f(n)$ when $(m, n) = 1$. It is elementary and well known that P and M form groups under $*$. We shall denote by δ the identity element for the operation $*$, so that $\delta(1) = 1$ and $\delta(n) = 0$ when $n > 1$. By f^{-1} is meant the inverse of f under $*$.

Another real commutative algebra is formed by A, as above, if the Dirichlet product $*$ is replaced by the unitary product $\times$ [3], [4]. The sets P and M also form groups under $\times$, and the identity element is again δ, the same as for $*$.

The first objective of the present paper is to show that the multiplicative groups formed by P and M, with either product, are isomorphic to the additive groups of the above algebras:

Theorem 1. *The groups* $\{P, *\}$, $\{M, *\}$, $\{P, \times\}$, $\{M, \times\}$ *and* $\{A, +\}$ *are all isomorphic.*

The proof of the isomorphism of $\{P, *\}$ and $\{A, +\}$ depends on the construction of a mapping $L : P \to A$ having the logarithmic property $L(f * g) = Lf + Lg$. We shall introduce and study such an operator L (the term operator referring to

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