A NOTE ON THE GENERALIZED HYPERGEOMETRIC DIFFERENTIAL EQUATION

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1. Introduction. We employ the usual notation

$$w' = {}_pF_{q-1}\left(\begin{matrix} a_p \\ 1 + b_{q-1} \end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_p)_k}{(1 + b_{q-1})_k} \cdot \frac{z^k}{k!} \tag{1.1}$$

for the generalized hypergeometric function, where $(a_p)_k$ is interpreted as $\prod_{r=1}^{p} (a_r)_k$, etc., and $(a)_k = a(a + 1) \cdots (a + k - 1)$, $(a)_0 = 1$.

It is common knowledge that ${}_pF_{q-1}$ is a solution of the differential equation of order max (p, q),

$$\left[z \frac{d}{dz} \prod_{s=1}^{q-1} \left(z \frac{d}{dz} + b_s\right) - z \prod_{s=1}^{p} \left(z \frac{d}{dz} + a_s\right)\right] w' = 0, \tag{1.2}$$

provided that no $1 + b_{q-1}$ is zero or a negative integer (for more details see any of [4], [5], or [6]). Apart from its theoretical significance, this remarkable equation leads to elegant proofs of certain properties of ${}_pF_{q-1}$ and plays an essential role in the theory of special functions.

A need is often felt to express (1.2) in the equivalent form

$$\sum_{k=0}^{\max(p,q)} u_k' z^k \frac{d^k w'}{dz^k} = 0, \tag{1.3}$$

where it is readily seen that the u_k''s depend on z and the $p + q - 1$ parameters a_p and b_{q-1}. The transformation from (1.2) to (1.3) is straightforward but, in most cases, entails a surprising amount of labor.

The purpose of this note is to study the transformation of the differential equation satisfied by the slightly more general function

$$w = z^{\alpha}(\mu + z)^{\beta} {}_pF_{q-1}\left(\begin{matrix} a_p \\ 1 + b_{q-1} \end{matrix}\middle| \lambda(\nu + z)^{\gamma}\right)$$

where α, β, μ, ν, λ, γ are arbitrary parameters, $\lambda, \gamma \neq 0$, to a form similar to (1.3). Matrix notation will bring order into a relatively involved situation, and explicit expressions for the coefficients u_k will be given.

We note that results obtained along the line of this paper have already been given by Fields and Luke [3] for polynomials of the form (1.1), while our u_k's are related, for special values of the parameters, to the set of functions $c_{k,i}$ defined in a paper by Carlitz [2].

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