TRIGONOMETRY IN A HYPERBOLIC SPACE

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1. Introduction. We consider a non-euclidean triangle in the Kähler manifold $(\mathbf{H}, M)$ furnished with the Bergman metric $M(\mathbf{H})$ on the hypersphere H in the complex euclidean space C^n of complex dimension n. (Throughout this paper n stands for any integer ≥ 2.) By a non-euclidean triangle we mean a geodesic triangle with the ordinary angle replaced by the analytic angle (see §2 for definition).

The main object of this paper is to study on a given non-euclidean triangle in $(\mathbf{H}, M)$ some basic trigonometric identities, such as the laws of sines and cosines which arise in hyperbolic geometry, and related problems. For a non-euclidean triangle in $(\mathbf{H}, M)$ a suitably modified law of sines holds while a law of cosines may not. The law of cosines holds if and only if one of the three angles of the non-euclidean triangle is equal to the corresponding ordinary angle (see §3). From this follows immediately a stronger form of Pythagorean theorem obtained in [2]. As was pointed out in [2] the following statement is not true in general: Through a point c in H not lying on a given geodesic γ_H there exists a geodesic which intersects γ_H at the right analytic angle. Theorem 4.1 gives such an example. A necessary and sufficient criterion that the above statement be true is given in Theorem 4.2. There are domains $\mathbf{D}$ in which the theorems in §3 fail to hold in $(\mathbf{D}, M)$. Such examples are exhibited in §5.

2. Preliminaries. We consider a bounded domain D in the complex euclidean space $\mathbf{C}^n$ of complex dimension n with the coordinates $z = (z^1, \cdots, z^n)$. The Bergman metric

$$M(\mathbf{D}): \qquad ds_{\mathbf{D}}^2 = T_{\alpha\bar{\beta}}\, dz^\alpha\, d\bar{z}^\beta$$

of $\mathbf{D}$ is kählerian and invariant under biholomorphic mappings of $\mathbf{D}$[1]; here we use the summation convention.

For any tangent vectors $u = (u^\alpha)$ and $v = (v^\alpha)$ at z in the Kähler manifold (D, M) we let

$$(2.1) \qquad [u, v] = T_{\alpha\bar{\beta}} u^\alpha \bar{v}^\beta, \; ||u||^2 = [u, u].$$

If two vectors u and v are independent in $\mathbf{C}^n(\mathbf{R})$, these vectors determine a plane section, $S(u, v) = [w = \lambda u + \mu v]$, where λ, μ are real parameters. Let $S_z(u, v)$ be a section defined by two tangent vectors u and v at z in $\mathbf{D}$. It is called *real* if $\mathrm{Im}\,[u, v] = 0$ at z and *analytic* if $v = \lambda u$ for some $\lambda \in \mathbf{C}$. The ordinary angle Θ between two vectors u and v at $z \in \mathbf{D}$ is given by

$$(2.2) \qquad \cos \Theta = |\mathrm{Re}\,[u, v]|/||u||\,||v||.$$

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