A DISK IN n-SPACE WHICH LIES ON NO 2-SPHERE

By R. C. Lacher

In a talk in 1959, Bing described a disk (=2-cell) in R^3 which is not a subset of any 2-sphere in R^3. Bing's and other examples are described by Martin in [3]. In [1], Bean produced an example of such a disk with only two wild points (and at the same time proved that a disk with only one wild point in R^3 lies on a 2-sphere in R^3). None of the constructions seem to generalize readily to higher dimensional euclidean spaces. In this note we introduce a new method of constructing such wild disks which generalizes easily. More precisely, we prove:

Theorem. *For each $n \geq 3$, there is a disk $D = D_n^2$ in R^n such that*

(1) *The set C of points of D at which D fails to be locally flat in R^n is a compact, zero-dimensional subset of the interior of D; and*
(2) *The boundary of D is homotopically essential in $R^n - C$.*

Corollary. *For $n \geq 3$, D_n^2 is a disk in R^n which is locally flat at each boundary point and which lies on no 2-sphere in R^n.*

The Cantor set C turns out to be the one constructed by Blankinship in [2]. In fact, to understand this proof, one must have some familiarity with Blankinship's paper. We recall the construction in [2] briefly.

$$C = \bigcap_{l=0}^{\infty} A_l ,$$

where $A_0 = T$ is a nice differentially embedded copy of $T^n = B^2 \times (S^1)^{n-2}$; A_1 is the union of a finite number of copies $T_1 , \cdots , T_k$ of T^n, the T_i being pairwise disjoint and "linked" in the interior of T; and A_l , $l \geq 2$, is as defined in the following paragraph.

There are (linear) homeomorphisms $f_i : T \approx T_i$, $i = 1, \cdots , k$. More generally, for each finite sequence $\alpha = (i_1 , \cdots , i_l)$ of integers with $1 \leq i_j \leq k$, define $\lambda(\alpha) = l$, and let

$$f_\alpha = f_{i_1} f_{i_2} \cdots f_{i_l} , \; T_\alpha = f_\alpha(T).$$

Then we can define A_l by

$$A_l = \bigcup_{\lambda(\alpha)=l} T_\alpha .$$

By describing the tori $T_1 , \cdots , T_k$ carefully, Blankinship was able to prove the following facts:

Received June 6, 1967. Partially supported by NSF Grant GP-5860.