A CONVOLUTION PRODUCT FOR THE SOLUTIONS OF PARTIAL DIFFERENCE EQUATIONS

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1. Introduction. The purpose of this paper is to establish a formula which associates with any two solutions of a partial difference equation with constant coefficients a new solution which is represented as a convolution product. This product is based upon a discrete analogue of Green's formula in the plane.

The lattice points of the complex plane are the points $z = m + ni$, where m and n may assume the values $0, \pm 1, \pm 2, \cdots$. Let $u(z)$ be a complex-valued function defined on the lattice points of the plane. The translation operators are defined as follows:

$$X^m u(z) = u(z+m); \qquad Y^n u(z) = u(z+in), \qquad m, n = 0, \pm 1, \pm 2, \cdots. \tag{1}$$

We are concerned with solutions of the partial difference equation

$$Lu(z) = 0, \tag{2}$$

where $L = \sum_{i=1}^{k} c_i T_i$, $T_i = X^{m_i} Y^{n_i}$, m_i, n_i are real integers and c_i are complex constants. Note that the operator $T_i^{-1} = X^{-m_i} Y^{-n_i}$ is the inverse of T_i.

Particular examples of the partial difference equations of concern are:

(3) $(X - 2I + X^{-1} + Y - 2I + Y^{-1})u(z) = 0$ (Laplace's equation)

(4) $[(X - 2I + X^{-1}) - c^{-2}(Y - 2I + Y^{-1})]u(z) = 0$ (Wave equation)

(5) $(X - 2I + X^{-1} + Y - 2I + Y^{-1})^2 u(z) = 0$ (Biharmonic equation)

(6) $[(X - 2I + X^{-1}) - c(Y - I)]u(z) = 0$ (Heat equation)

(7) $(I + iX - XY - iY)u(z) = 0$ (Cauchy-Riemann equation, complex form)

A problem of interest concerning such difference equations is the generation of new solutions from a given solution. One approach to this problem was given by Duffin and Shelly [2] by the definition of operators under which the solution set is invariant. A new class of such operators is studied here.

The first four of the above equations are self-explanatory; the last refers to the theory of discrete analytic functions, and solutions of this equation are termed discrete analytic. It was shown by Duffin and Duris [1] that, given two solutions $w(z)$ and $u(z)$ of (7) there is a new solution $\Phi(z) = w * u$, where "*" was termed a convolution product. This product is both commutative and associative. In this paper we also introduce a convolution product for solutions

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