# STRENGTHENING ALEXANDER'S SUBBASE THEOREM

By G. E. Strecker, E. Wattel, H. Herrlich, and J. de Groot

Historically, compactness has been introduced within the framework of topology. However, it is illuminating and convenient to define compactness set-theoretically.

Let $X$ be a non-empty set and $E$ a family of subsets of $X$. If $A \subset X$, then $A$ will be said to be *compact relative to* $\mathfrak{E}$, or equivalently $A \in \rho\mathfrak{E}$, provided that for every $\mathfrak{B} \subset \mathfrak{E}$ such that $\mathfrak{B} \cup \{A\}$ has f.i.p., $(\bigcap \mathfrak{B}) \cap A \neq \varnothing$. (Although compactness is usually defined in terms of coverings (by open sets), we prefer, for reasons of simplicity and convenience, to work within the complementary framework of collections with the finite intersection property (f.i.p.).) Thus in the case that $\mathfrak{E}$ is the collection of all closed subsets of a topological space, denoted $(X, \mathfrak{E})$, $\rho\mathfrak{E}$ is by this definition the collection of all compact subsets of the space. Observe that $\rho\mathfrak{E}$ contains all finite subsets of $X$, but $\rho\mathfrak{E}$ need not contain $\mathfrak{E}$. $\rho^n\mathfrak{E}$ is defined inductively; $\rho^n\mathfrak{E} = \rho(\rho^{n-1}\mathfrak{E})$. Furthermore, we let $\gamma\mathfrak{E}$ denote the collection of all (arbitrary) intersections of finite unions of members of $\mathfrak{E}$. Observe that $\gamma$ is idempotent; $\gamma^2\mathfrak{E} = \gamma\mathfrak{E} \supset \mathfrak{E}$. Also the convention $\bigcap\varnothing = X$ is used. Thus $\mathfrak{E} = \gamma\mathfrak{E}$ if and only if $(X, \mathfrak{E})$ is a topological space.

In terms of these operators, Alexander's Subbase Theorem can be stated as follows:

Theorem (Alexander). *For every $\mathfrak{E} \subset 2^X$, $\rho\mathfrak{E} = \rho\gamma\mathfrak{E}$; i.e. the family of sets compact relative to $\mathfrak{E}$ is the same as the family of sets compact relative to the larger collection $\gamma\mathfrak{E}$.*

In the course of the paper the theorem is strengthened by establishing the existence of an even larger collection, namely $\gamma(\mathfrak{E} \cup \rho^2\mathfrak{E})$, with the same compact sets. Also, necessary and sufficient conditions are obtained which determine whether or not $\gamma\mathfrak{E}$ is the largest collection $\mathfrak{D}$ for which $\rho\mathfrak{E} = \rho\mathfrak{D}$, or indeed whether or not there exists a collection $\mathfrak{D}$ maximal with respect to the property that $\rho\mathfrak{E} = \rho\mathfrak{D}$. (As usual, the term "largest" implies comparability with all other elements, whereas "maximal" does not necessarily carry that connotation.) For a Hausdorff space, $(X, \gamma\mathfrak{E})$, there is always a maximal collection—precisely $\gamma(\mathfrak{E} \cup \rho^2\mathfrak{E})$, and $\gamma\mathfrak{E}$ is maximal if and only if $(X, \gamma\mathfrak{E})$ is a $k$-space.

**1. An extension of Alexander's Theorem.** Throughout the paper we will assume that $X$ is a non-empty set and that $\mathfrak{E}$ and $\mathfrak{D}$ are subsets of $2^X$. $\mathfrak{E} \wedge \mathfrak{D}$ will denote $\{E \cap D \mid E \in \mathfrak{E}, D \in \mathfrak{D}\}$.

Lemma 1. *If $\mathfrak{E} \subset \mathfrak{D}$, then $\rho\mathfrak{D} \subset \rho\mathfrak{E}$.*

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