GENERALIZED CENTRALIZERS OF MODULES

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1. Introduction. Assume R is a ring and M is a nonzero (right) R-module. The collection τ of large submodules of M has the following properties:

(a′) if $T \in \tau$ and $T \subseteq N$ then $N \in \tau$,

(b′) if $S, T, \in \tau$ then $S \cap T \in \tau$,

(c′) if $S, T \in \tau$ and $\alpha: T \to M$ is a module map such that $\alpha^{-1}(S) = U$, then $U \in \tau$.

The dual notion of *large* submodule is *small* submodule. A submodule T is small if $T + N = M$ implies $N = M$. The collection of small submodules satisfies conditions (a), (b), and (c) which are dual to (a′), (b′), and (c′). We note that the concepts *large* and *small* are called *essential* and *superfluous* by some authors.

In this paper we show how to associate with each system τ of submodules of M satisfying either conditions (a), (b), (c) or their duals a ring which we call a *generalized centralizer* of M. In the special case where τ is the collection of all large submodules the associated ring is the *extended centralizer* introduced by R. E. Johnson in [2].

Whenever the module M is complemented in a certain way with respect to τ, then the associated generalized centralizer is a regular ring. M is always complemented in this way when τ is the collection of large submodules.

In §3 a number of examples of modules with systems τ satisfying (a′), (b′), and (c′) or their duals are given.

In §4 a more extended treatment is given to the case where τ is the collection of all small submodules. Here it is shown that if M has a certain weak quasi-projectivity property, then the associated generalized centralizer is a (Jacobson) semisimple ring.

In certain important cases, $\mathrm{Hom}_R(M, M)$ is a subring of the generalized centralizer of $\hat{M}/M$, where $\hat{M}$ is the injective hull of M.

2. We assume that R is a ring and M is a right R-module. If $N_1 \subseteq N_2$ are submodules of M then $\pi: M/N_1 \to M/N_2$ will denote the natural map, $\pi(m + N_1) = m + N_2$, $m \in M$.

Let τ denote a non-empty collection of proper submodules of M such that

(a) if $T \in \tau$ and $N \subseteq T$, then $N \in \tau$,

(b) if $S, T, \in \tau$ then $S + T \in \tau$,

(c) if $S, T \in \tau$ and $\alpha: M \to M/T$ is a module map such that $\alpha(S) = U/T$, then $U \in \tau$.

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