ORDERED EXTENSIONS OF ORDERED RINGS

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An extension of a ring A by another ring Λ is an exact sequence

$$0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} \Lambda \to 0$$

of rings and ring homomorphisms. We are concerned here with the problem of making E into a partially ordered ring when both A and Λ are partially ordered rings, in a natural way relative to the given orders on A and Λ. The corresponding problem for lattice-ordered rings is also studied.

1. **Preliminaries.** A ring A is called a *partially ordered ring* if it has a set of nonnegative elements, satisfying the conditions: (i) $a \geq 0$, $b \geq 0$ implies $a + b \geq 0$ and $ab \geq 0$; (ii) $a \geq 0$, $-a \geq 0$ if and only if $a = 0$. The symbol $a \geq b$ then is defined to mean $a - b \geq 0$. We shall use the shorter expression "ordered ring". If A is in addition a lattice, it is called a *lattice-ordered ring*. To prove that an ordered ring A is a lattice-ordered ring, it suffices to establish the existence of $a \vee b$ for every a and b in A [1, 0.19]. In a lattice-ordered ring, $|a|$ denotes the element $a \vee -a$; it satisfies $|a| \geq 0$ [1, 5A]. An ideal I in an ordered ring A is said to be *convex* if whenever $0 \leq a \leq b$ and $b \in I$, then $a \in I$. An ideal I in a lattice-ordered ring is said to be *absolutely convex* if, whenever $|a| \leq |b|$ and $b \in I$, then $a \in I$. Clearly, an absolutely convex ideal is convex. It is easy to see that if I is a convex ideal in a lattice-ordered ring, then $|a| \in I$ implies $a \in I$.

The following statements are proved in [1, 5.2 and 5.3]. Here $I(a)$ denotes the image of a under the canonical homomorphism of A onto A/I.

(1) Let I be an ideal in an ordered ring A. In order that A/I be an ordered ring, according to the definition:

$I(a) \geq 0$ if and only if there exists $b \in A$ such that $b \geq 0$ and $I(b) = I(a)$, it is necessary and sufficient that I be convex.

(2) The following conditions on a convex ideal I in a lattice-ordered ring A are equivalent:

(i) I is absolutely convex.
(ii) $a \in I$ implies $|a| \in I$.
(iii) $I(a \vee b) = I(a) \vee I(b)$.

The induced order on a quotient ring will always mean the order as defined in (1). The canonical homomorphism of A onto A/I is then order-preserving. More generally, any epimorphism $\beta : A \to B$ with convex kernel induces an order on B that makes it an ordered ring, since B can be identified with a quotient ring.

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