ON KUZMIN'S THEOREM, II

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Let α be a number for which $0 < \alpha < 1$. Put

$$\alpha = [0; a_1, a_2, \cdots], \tag{1}$$

i.e. let $a_1, a_2, \cdots$ be the partial quotients of the regular continued fraction expansion of α; further put

$$\theta_n = [a_n; a_{n+1}, \cdots], \tag{2}$$

$$m_n(x) = |E(\alpha: \theta_n^{-1} \leq x)|, \tag{3}$$

where x is a real number between 0 and 1. ($E(\cdots)$ means the set of the numbers in (0, 1) having the property indicated in the brackets and $|E|$ is the Lebesgue measure of E.) Gauss wrote in a letter to Laplace that he succeeded in proving

$$\lim_{n\to\infty} m_n(x) = \frac{\log(1+x)}{\log 2}, \tag{4}$$

(of course Gauss did not speak of Lebesgue-measure but formulated (4) in the language of probability) but his proof is not known. The first proof of (4) is due to R. O. Kuzmin [1] who proved the sharper result

$$m_n(x) = \frac{\log(1+x)}{\log 2} + O(q_1^{\sqrt{n}}), \tag{5}$$

where $q_1 < 1$; one year later without having known of the work of Kuzmin, P. Lévy [2] proved the still sharper result

$$m_n(x) = \frac{\log(1+x)}{\log 2} + O(q_2^n) \qquad (q_2 < 1). \tag{6}$$

P. Lévy's proof is based on an idea quite different from that of Kuzmin.

A few years ago, using Kuzmin's method, the present author [3] proved (6) in a very simple way and could even replace P. Lévy's q_2 by a smaller one. Kuzmin's proof of (5) was based on the following recursion formula

$$m_{n+1}(x) = \sum_{k=1}^{\infty} \left\{ m_n\left(\frac{1}{k}\right) - m_n\left(\frac{1}{k+x}\right) \right\}. \tag{7}$$

((7) is obvious because of $\theta_n = a_n + 1/(\theta_{n+1})$).

Now (6) was contained as a special case of the following theorem:

Theorem 1. *Let $f_0(x)$ be twice continuously differentiable, $f_0(0) = 0$, $f_0(1) = 1$.*

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