SOME BEST POLYNOMIAL APPROXIMATIONS IN THE PLANE

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If B is a compact point set in the complex plane and $f(z)$ a given function continuous on B, there exists a unique polynomial $p_n^* \in P_n$, P_n being the set of polynomials of degree at most n, such that

$$\|f - p_n^*\| < \|f - p\| \tag{1}$$

for all $p \in P_n$, $p \neq p_n^*$, where the norm in question is the uniform norm on B. We call p_n^* the best approximation to f (on B out of P_n), and write $E_n(f; B) = \|f - p_n^*\|$. Our main object here is to present several classes of examples of both analytic and nonanalytic functions where the best approximation can be determined explicitly for all n. In the first section we present the general theory of best approximations. The second section is devoted to some old and new exact polynomial approximations to analytic functions, while the third section is devoted to approximations to a special class of nonanalytic but continuous functions on the disc.

1. A characterization of best approximations. We begin by recalling some characteristic properties of best approximations. A signature, Σ, is a finite set of ordered pairs of complex numbers $\{(z_i, \epsilon_i)\}_{i=1}^r$ where the $z_i \in B$ are distinct and $|\epsilon_i| = 1$. (Occasionally, we refer to the set $\{z_i\}$ as Σ.) Σ is said to be *extremal* for approximation out of P_n if there exist nonzero complex numbers $\zeta_1, \cdots, \zeta_r$ such that

$$\operatorname{sgn} \zeta_j = \bar{\epsilon}_j, \qquad j = 1, \cdots, r \tag{2}$$

and

$$\sum_{j=1}^{r} \zeta_j p(z_j) = 0 \tag{3}$$

for all $p \in P_n$. It is clear that if Σ is an extremal signature, then $r \geq n + 2$.

If $g \in C(B)$, we put

$$E(g) = \{z \in B / |g(z)| = \|g\|\},$$

i.e., E is the set where g assumes its maximum value on B. We now quote a result which characterizes best approximations.

Theorem A. (Rivlin and Shapiro [3]) *If $p \in P_n$, Σ is an extremal signature for P_n and $\Sigma \subset E(f - p)$ with* $\operatorname{sgn}(f(z_i) - p(z_i)) = \epsilon_i$, $i = 1, \cdots, r$, *then p is a best approximation to f. Conversely, if p^* is a best approximation to f on B*

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