SEMI-DENSE CLOSURE OF SYSTEMS OF FUNCTIONS

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1. Introduction. A set of elements $\{x_\alpha\}$ of a normed linear space X is called *closed* if any element of the space X can be approximated as closely as desired by finite linear combinations of elements of the set. For example, the Weierstrass Approximation Theorem tells us that the powers $1, x, x^2, \cdots$ are closed in the normed linear space $C[a, b]$, $\|f\| = \max_{a \le x \le b} |f(x)|$, $-\infty < a < b < \infty$. Some sets are so plentifully provided with elements that any infinite subset will still be closed. An example of this is the set of functions $x^{1/n}$ $(n = 1, 2, \cdots)$. By Müntz' Theorem, any infinite subset of these functions is closed in $L^2[0, 1]$. A closed set with this property has been called "densely-closed" (dicht-abgeschlossen), see Kacmarz and Steinhaus [9; 53]. At the other end of the road, there are, of course, closed sets which cease to be closed as soon as a single element is omitted. Such sets are called *minimally closed.* For example, any closed orthonormal system in a Hilbert space is minimally closed.

The object of the present paper is to study a situation that lies in between: closed sets which remain closed after any finite number of elements have been discarded. Such a set will be called *semi-densely closed.* We shall give a sufficient condition for semi-dense closure as well as several specific examples and applications of the concept.

2. A sufficient condition for semi-dense closure. Let us recall that a set of polynomials $\{p_n\}$ $n = 0, \cdots$ is a *basic set* if every polynomial q has a *unique* representation as a finite combination of p's:

$$q = \sum_{k=0}^{J(q)} c_k p_k \,. \tag{2.1}$$

The degree of p_n need not be n, but in many familiar instances it is in fact n. With every basic set of polynomials there can be associated a biorthonormal set of linear functionals $\{\mathcal{L}_n\}$:

$$\mathcal{L}_m(p_n) = \delta_{mn} \,, \tag{2.2}$$

and a formal expansion of a function f in a so-called basic series

$$f \sim \sum_{k=0}^{\infty} \mathcal{L}_k(f) p_k \,. \tag{2.3}$$

For any polynomial q, the basic series expansion

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