ON THE RATE OF GROWTH OF TYPICALLY REAL FUNCTIONS

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1. Introduction. A function $f(z)$ given by

$$f(z) = z + a_2 z^2 + \cdots \tag{1.1}$$

which is analytic in $|z| < 1$ is said to be typically real if $\operatorname{Im} f(z) = 0$ if and only if $\operatorname{Im} z = 0$. This class of functions, which we denote by T, was introduced by Rogosinski [10]. A sufficient condition for a function to be typically real is that $f(z)$ be univalent and have real Taylor coefficients. Thus T contains an important subclass of the class S of functions given by (1.1) which are analytic and univalent in $|z| < 1$.

Robertson [8] has shown that $f(z) \in T$ if and only if

$$f(z) = \int_0^\pi K(z, t)\, d\alpha(t) \tag{1.2}$$

where $K(z, t) = z(1 - 2z \cos t + z^2)^{-1}$, $\alpha(t)$ is increasing on $[0, \pi]$ and $\alpha(\pi) - \alpha(0) = 1$. In this paper we determine necessary and sufficient conditions depending on $\alpha(t)$ in order that for fixed θ, $f(re^{i\theta}) = O(1 - r)^{-\lambda}$ and that $M(r, f) = O(1 - r)^{-\lambda}$ where

$$M(r, f) = \max_{|z|=r} |f(z)|.$$

As applications of these results we study the relation between the rate of growth of $M(r, f)$ and $f(\pm r)$, and the rate of growth of $L(r, f) = \int_0^{2\pi} r \mid f'(re^{i\theta})| \, d\theta$. The estimates on $L(r, f)$ enable us, using a standard technique, to estimate the Taylor coefficients, a_n , of a typically real function subject to the growth condition $f(\pm r) = O(1 - r)^{-\lambda}$ (to be read here and in the sequel as $f(r) = O(1 - r)^{-\lambda}$ and $f(-r) = O(1 - r)^{-\lambda}$). Throughout this paper A will denote a constant independent of the parameters involved in the given expression, but not necessarily the same constant in each case. A (θ) will denote a constant depending on the parameter θ, but independent of the other parameters in the given expression.

2. Main theorem.

Theorem 2.1. *If $f(z) \in T$, is given by* (1.2) *and* $0 < \lambda \leq 2$, *then* $M(r, f) = O(1 - r)^{-\lambda}$ *if and only if* $\alpha(t)$ *satisfies the conditions:*

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