RADIAL SYMMETRIZATION AND CAPACITIES IN SPACE

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1. Introduction. Let E^n be finite Euclidean n-space, $n \geq 3$. A *ring* R is a domain in E^n whose complement consists of a bounded component C_0 and an unbounded component C_1. Given a number α, $2 \leq \alpha \leq n$, we define the α-*capacity* of a ring R as follows:

$$C_\alpha(R) = \inf_u \int_R |\nabla u|^\alpha \, d\omega,$$

where the infimum is taken over all functions $u(x)$, which are continuously differentiable in R and have boundary values 0 on *bdry* C_0 and 1 on *bdry* C_1. The n-capacity of a ring in E^n is the *conformal capacity*. Gehring [2] proved that the conformal capacity of a ring in 3-space is not increased by either spherical or point symmetrization. Similar results for Steiner and Schwarz symmetrization have been obtained by Anderson [1].

In this paper we define a class of radial symmetrization methods for domains in E^n and show that the α-capacity of a ring is not increased by the appropriate radial symmetrization. These methods are natural extensions of symmetrization processes introduced by Szegö [8] and Marcus [5].

Suppose u is the extremal function for the conformal capacity of a ring R. We show (Theorem 2) that the sets where $u \leq a$ $(0 < a < 1)$ are starlike sets if both C_0 and $R \cup C_0$ are starlike. A similar result is established for the α-capacity. Stoddart [7] proved this for harmonic functions in E^3.

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2. Preliminaries. Given a set S in E^n we let $\mathbf{C}S$ and $m_k(S)$ denote the complement, and k-dimensional Lebesgue measure of S, respectively. Points in E^n will be designated by capital letters P, Q or by small letters x and y with rectangular coordinates $x = (x_1, \cdots, x_n)$. Points will be treated as vectors with norm and inner product denoted by $|x|$ and $(x \cdot y)$, respectively.

It will be convenient to use the spherical coordinates $r, \varphi_1, \cdots, \varphi_{n-1}$ of a point $x = (x_1, \cdots, x_n)$ where $r = |x|$, $0 \leq \varphi_{n-1} < 2\pi$, $0 \leq \varphi_k \leq \pi$ $(k = 1, \cdots, n-2)$, and $x_1 = r \cos \varphi_1$, $x_k = r \sin \varphi_1 \cdots \sin \varphi_{k-1} \cos \varphi_k$ $(k = 2, \cdots, n-2)$, $x_{n-1} = r \sin \varphi_1 \cdots \sin \varphi_{n-2} \cos \varphi_{n-1}$, $x_n = r \sin \varphi_1 \cdots \sin \varphi_{n-1}$. Given numbers φ_k' $(1 \leq k \leq n-1)$ we let $L(\varphi')$ denote the *ray* $L(\varphi') = \{x : 0 \leq |x|, \varphi_k = \varphi_k', 1 \leq k \leq n-1\}$. The intersection $D \cap L(\varphi)$ of a set D with the ray $L(\varphi)$ will be denoted by $D(\varphi)$.

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