ON A SPACE OF FUNCTIONS OF WIENER

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1. Introduction. In the formulation of one of his tauberian theorems [3;27], Wiener introduced continuous functions f on $(-\infty, \infty)$ for which $\sum_{k=-\infty}^{\infty} \max_{k \le x \le k+1} |f(x)|$ converges. In this paper we give a systematic analysis of the space of all such functions normed with this sum. The space turns out to be a Banach algebra (with convolution as multiplication) and is a direct sum of two subspaces, one of which is (equivalent to) the sequence space l^1. Knowledge of this structure enables us to find all continuous linear functionals on the space. This in turn allows us to formulate the tauberian theorem mentioned above in a way that permits a quick proof via distribution theory modelled on a proof of Korevaar of the "famous" tauberian theorem. Finally, we discuss the analogous space of functions on the plane which differs from the case of the line in some respects.

2. The structure of the space T. For $k = 0, \pm 1, \pm 2, \ldots$ let I_k denote the closed interval $[k, k+1]$. Let T be the linear space of all continuous f on $(-\infty, \infty)$ for which

$$||f||_T = \sum_{k=-\infty}^{\infty} \max_{x \in I_k} |f(x)|$$

is finite. It is readily verified that $||\cdot||_T$ is a norm. For $f \in T$ we have

$$\int_{-\infty}^{\infty} |f| = \sum_{k=-\infty}^{\infty} \int_{I_k} |f| \le \sum_{k=-\infty}^{\infty} \max_{x \in I_k} |f(x)| = ||f||_T .$$

Hence $T \subset L^1(-\infty, \infty)$ and $||f||_1 \le ||f||_T$ for $f \in T$. Also it is clear that if $f \in T$, then f vanishes at infinity.

Theorem A. *The normed linear space T is complete. That is, T is a Banach space.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in T. Then given $\epsilon > 0$ there exists N such that if $m, n \ge N$, then

$$(1) \qquad ||f_m - f_n||_T = \sum_{k=-\infty}^{\infty} \max_{x \in I_k} |f_m(x) - f_n(x)| < \epsilon.$$

Hence, for any fixed k,

$$\max_{x \in I_k} |f_m(x) - f_n(x)| < \epsilon$$

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