# ON RICCI CURVATURE AND GEODESICS

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**1. Introduction.** The relationship in a Riemannian manifold between the numerical invariants of curvature at each point and the global metric properties of the manifold has been for a long time a subject for study by many people. In 1941 S. B. Myers [4], extending an earlier result by J. L. Synge, proved that, if the Ricci (mean) curvature of a manifold is uniformly bounded from below by a positive constant $c^2$, then every geodesic arc of length $\geq \pi/c$ contains conjugate points; in particular, if the manifold is complete, its diameter is bounded by $\pi/c$. A qualitative generalization of this result was published by W. Ambrose [1] in 1957, his statement reading as follows: Let $\mathbf{M}$ be a complete, $n$-dimensional Riemannian manifold ($n \geq 2$); suppose that there exists a point $\mathbf{p}_0 \in \mathbf{M}$ with the property that, for every infinite geodesic ray $\Gamma$ issued from $\mathbf{p}_0$ and parametrized by its arc length $s$, the following improper integral diverges,

$$\liminf_{a \to \infty} \int_0^a K(s)\, ds = +\infty,$$

where $K(s)$ is the Ricci curvature at the tangent vector of $\Gamma$ at the point corresponding to arc length $s$; then each $\Gamma$ contains (infinitely many) conjugate points of $\mathbf{p}_0$, and consequently $\mathbf{M}$ is compact.

The present article contains a different sort of extension of Myers' theorem; the following statement arises as a consequence of the main result, which is stated as Theorem 2 in the next paragraph.

Theorem 1. *Let* $\mathbf{M}$ *be a complete, n-dimensional Riemannian manifold with nonnegative Ricci curvature everywhere. If, for some point* $\mathbf{p}_0 \in \mathbf{M}$, *every geodesic ray* $\Gamma$ *issuing from* $\mathbf{p}_0$ *has the property that*

$$\limsup_{a \to \infty} \left\{ \int_0^a \sqrt{K(s)}\, ds - \tfrac{1}{2} \log a \right\} = \infty, \tag{1.1}$$

*then $M$ is compact; $K(s)$ denotes the Ricci mean curvature of* $\mathbf{M}$ *at the tangent vector of* $\Gamma$ *over the point at arc distance $s$ from* $\mathbf{p}_0$.

The above theorem and Ambrose's earlier one contain the same conclusion under two apparently similar assumptions; neither of these two sets of assumptions contains the other, so that these two theorems can be used jointly as independently sufficient conditions for compactness. The integral (1.1) over geodesic arcs without conjugate points, however, is of independent interest; a sharp estimate of such an integral has been used implicitly before [3, proof of

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