COMPACT AND RELATED MAPPINGS

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1. Introduction. A mapping (continuous function) $f : X \rightarrow f(X) = Y$ from one topological space onto another is *compact* if for every compact set $K \subset Y$, $f^{-1}(K)$ is compact. A point p in Y is a *singular point* of f (with respect to compactness) if contained in every neighborhood of p there is a compact set K such that $f^{-1}(K)$ is not compact.

In this paper, we study the relationships between compactness of a mapping and certain other common mapping properties (e.g., closed, quasi-open) by investigating properties of the set S of singular points and its inverse image.

2. The Whyburn unified space of a mapping. We shall make use of the notion of the unified space of a mapping. This section contains results related to this important concept that will be used in the sequel.

Given a mapping $f : X \rightarrow f(X) = Y$ from one Hausdorff space onto another, G. T. Whyburn [6] defines a new "unified space" $Z = X' + Y'$ consisting of a point $x' = h(x)$ for each x in X and a point $y' = k(y)$ for each y in Y so that $h(X) = X'$, $k(Y) = Y'$, h and k are one-to-one, and $X' \cdot Y' = \phi$.

A subset $Q \subset Z$ is defined to be open provided the following three conditions hold:

(i) $h^{-1}(Q \cdot X')$ is open in X.

(ii) $k^{-1}(Q \cdot Y')$ is open in Y.

(iii) For every compact set $K \subset k^{-1}(Q \cdot Y')$,

$$f^{-1}(K) \cdot [X - h^{-1}(Q \cdot X')] \text{ is compact.}$$

In [6], Whyburn established that the collection of all such open sets Q is a topology for Z and that Z is a separable metric space if X and Y are separable metric and locally compact. We shall use the facts, also proved in [6], that h is strongly open and k is strongly closed (thus h and k are homeomorphisms), and that the function $r : Z \rightarrow r(Z) = Y'$ defined by $r(z) = z$ if $z \in Y'$ and $r(z) = k(f(h^{-1}(z)))$ if $z \in X'$ is continuous.

(2.1) Theorem. *A point p in Y is a singular point of f if and only if $k(p)$ is an accumulation point of X'.*

Proof. First suppose there is an open neighborhood $U \subset Z$ of $p' = k(p)$ such that $U \cdot X' = \phi$. Then $X - h^{-1}(U \cdot X') = X$, so that if K is any compact subset of $k^{-1}(U)$, then $f^{-1}(K) \cdot [X - h^{-1}(U \cdot X')] = f^{-1}(K)$ is compact.

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