ORTHOGONAL POLYNOMIALS OF HYPERGEOMETRIC TYPE

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1. Introduction. The generalized hypergeometric polynomial of degree n is defined by means of

$$(1.1)\qquad {}_{p+1}F_q\left[\begin{matrix}-n, \alpha_1, \alpha_2, \cdots, \alpha_p; \\ \beta_1, \beta_2, \cdots, \beta_q;\end{matrix}\ x\right] = \sum_{r=0}^{n} \frac{(-n)_r(\alpha_1)_r \cdots (\alpha_p)_r}{r!\,(\beta_1)_r \cdots (\beta_q)_r} x^r,$$

where

$$(\alpha)_r = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+r-1), \qquad (\alpha)_0 = 1.$$

Most of the well-known orthogonal polynomials are hypergeometric polynomials of various types. For example, the Jacobi polynomial $\{P_n^{(\alpha,\beta)}(x)\}$ is defined by means of [5; 254]

$$(1.2)\qquad P_n^{(\alpha,\beta)}(x) = \frac{1}{n!}(1+\alpha)_n\ {}_2F_1\left[\begin{matrix}-n, n+\alpha+\beta+1; \\ \alpha+1;\end{matrix}\ \tfrac{1}{2}(1-x)\right].$$

Similarly the Laguerre polynomial $\{L_n^{(\alpha)}(x)\}$ is

$$(1.3)\qquad L_n^{(\alpha)}(x) = \frac{1}{n!}(1+\alpha)_n\ {}_1F_1\left[\begin{matrix}-n; \\ \alpha+1;\end{matrix}\ x\right].$$

Another set of orthogonal polynomials is the set of Bessel polynomials defined by Krall and Frink. In the notation of Al-Salam [2], we put

$$(1.4)\qquad Y_n^{(\alpha)}(x) = {}_2F_0\left[\begin{matrix}-n, n+\alpha+1; \\ -;\end{matrix}\ -\tfrac{1}{2}x\right].$$

The question which we raise here is whether there exist other classes of orthogonal polynomials of hypergeometric type in addition to the known ones.

It has been proved by the writer and W. A. Al-Salam [1], that the Laguerre polynomials are the only hypergeometric polynomials of the form

$${}_{p+1}F_q\left[\begin{matrix}-n, \alpha_1, \alpha_2, \cdots, \alpha_p; \\ \beta_1, \beta_2, \cdots, \beta_q;\end{matrix}\ x\right],$$

where n is a non-negative integer and α_i, β_i are independent of x and n.

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