DIFFERENTIABLE MANIFOLDS IN COMPLEX EUCLIDEAN SPACE

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We consider a k-dimensional differentiable submanifold M^k of n-dimensional complex coordinate space $\mathbf{C}^n$, and interest ourselves in the relation of the submanifold M to the analytic subvarieties of $\mathbf{C}^n$.

Consider the tangent variety P to M at a point p in M. It is a k-dimensional real-linear variety and if $k \geq n$, it will contain a complex-linear variety of complex dimension $k - n$. The point p will be called *exceptional* if P contains a complex-linear variety of dimension $k - n + 1$.

We would like information about the structure of the set of exceptional points and about the structure of M in the neighborhood of an exceptional point. We would also like information about the submanifolds of M which bound analytic subvarieties of $\mathbf{C}^n$. We would like to use this information to describe the polynomially convex hull and the hull of holomorphy of M, and perhaps eventually to give a satisfactory explanation of some of the known results [1] on the cohomology groups of polynomially convex sets in $\mathbf{C}^n$.

These problems seem to be very difficult. At least it is hard to prove global results. Therefore in this paper we consider primarily the local situation. Our only global result, having to do with the exceptional points of a two sphere imbedded in $\mathbf{C}^2$, is a consequence of a theorem of Chern and Spanier [2].

In case $M^2 \subset \mathbf{C}^2$, as instances of our problems there arise questions in one complex variable which seem not to have been considered to any extent, although from our point of view they are very natural. Here is an example. Let f be a continuous complex-valued function on $\mathbf{C}^1$. What can be said about the family of all simple closed curves $\gamma \subset \mathbf{C}^1$ such that f agrees on γ with the boundary values of some analytic function defined on the interior of γ?

Although we get only the above-mentioned global result, we are able to obtain quite a bit of local information at a point p of M. We first investigate the case $k > n$, and put the equations of M in a simple form, by means of a differentiable coordinate transformation on M at p and an analytic coordinate transformation on $\mathbf{C}^n$ at p. The form of the simplified equations for M suggests the existence of certain families of simple closed curves on M each of which bounds an analytic disk in $\mathbf{C}^n$. By an iteration argument we prove the conjectured families indeed exist. It is not clear whether locally the union of the corresponding disks gives the complete hull of holomorphy of M in some appropriate sense.

We next consider an exceptional point p of $M^k \subset \mathbf{C}^k$, one which is not exceptionally exceptional. Such points are of two types, depending on whether

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