A FOURIER SERIES METHOD FOR ENTIRE FUNCTIONS

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If $f(z)$ is an entire function, and we let

$$c_p(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\log | f(re^{i\theta}) |)e^{-ip\theta}\, d\theta$$

be the p-th Fourier coefficient of $\log | f(re^{i\theta}) |$, then the behaviour of $| f(z) |$ is reflected in the behaviour of the sequence $\{c_p(r)\}$ and vice versa. We develop this idea here. The main result is Theorem 1; the preliminary results are standard, and the later results are easy corollaries of Theorem 1. A special case of Theorem 1 is that $f(z)$ is of exponential type if and only if for some constant M and all $r \geq 1$,

$$| c_p(r) | \leq \frac{Mr}{| p | + 1}, \qquad p = 0, \pm 1, \pm 2, \cdots .$$

Fourier series methods then yield estimates for some integrals involving $| f(z) |$. These estimates can be considered as minimum modulus theorems. For example, if f is of exponential type, then there is a positive constant α such that

$$\sup_{r\geq 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{| f(re^{i\theta}) |^{\alpha/r}}\, d\theta < 2.$$

And we prove, among other results, that f is of exponential type if and only if $r^{-1} \log | f(re^{i\theta}) |$ is uniformly in $L_2(-\pi, \pi)$ for $r \geq 1$. These results would seem to be difficult to prove by the standard arguments via canonical products. We make no use of canonical products. Finally, we obtain a new and informative proof of a theorem of Lindelöf that characterizes the zeros of entire functions of given order and type.

The following result was proved in [1]. We give a different proof.

Lemma 1. *If $f(z)$ is holomorphic in $| z | \leq r$, with zeros $z_n = r_n e^{i\theta_n}$, $f(0) \neq 0$, and $\log f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ near $z = 0$, then*

(1) $$\log | f(re^{i\theta}) | = \sum_{p=-\infty}^{\infty} c_p(r) e^{ip\theta}, \quad \textit{where the } c_p \textit{ are given by}$$

(2) $$c_0 = \sum_{r_n \leq r} \log \frac{r}{r_n} + \log | f(0) |$$

(3) $$c_p = \tfrac{1}{2}\alpha_p r^p + \frac{1}{2p}\sum_{r_n\leq r}\left(\frac{r}{z_n}\right)^p - \frac{1}{2p}\sum_{r_n\leq r}\left(\frac{\bar{z}_n}{r}\right)^p \quad \text{for } p > 0$$

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