ORDER IDEALS IN A C*-ALGEBRA AND ITS DUAL

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1. **Introduction.** J. Dixmier has obtained an elegant and useful characterization of the positive part of a two-sided ideal in a von Neumann algebra [3; 10–13]. In this paper we shall investigate the positive parts of 1) left ideals in a C^*-algebra, and 2) left invariant subspaces of the dual of a C^*-algebra. A subspace $\mathcal{L}$ of the dual $\mathfrak{A}^*$ of a C^*-algebra $\mathfrak{A}$ is *left invariant* if for all f in $\mathcal{L}$ and A in $\mathfrak{A}$, the function Af is also in $\mathcal{L}$, where Af is defined by $(Af)(B) = f(AB)$ for all B in $\mathfrak{A}$. The discussion will be primarily limited to ideals and subspaces that are closed in various topologies.

Two techniques have greatly simplified our work. The first is the recognition by Takeda [11] and Grothendieck [7] that the double dual of a C^*-algebra is a von Neumann algebra. This enables one to solve various problems for C^*-algebras with von Neumann algebra methods (e.g., see the new proof in §2 that closed two-sided ideals are self-adjoint). This approach favors a more general study of left invariance. The dual $\mathfrak{A}^*$ of a C^*-algebra $\mathfrak{A}$ coincides with the pre-dual of $\mathfrak{A}^{**}$ (the Banach space of ultra-weakly continuous linear functions on $\mathfrak{A}^{**}$), hence we are led to consider the problems associated with left-invariance in the pre-dual of a von Neumann algebra. We are indebted to R. V. Kadison for introducing us to the articles of Takeda and Grothendieck.

The second important technique that we have used is Tomita's polar decomposition for elements in the dual of a C^*-algebra [13]. As we have indicated in the text, some of our results are reformulations of those of Tomita.

When closed in a suitable topology, the positive part of a left ideal or a left invariant subspace is an *order ideal.* An order ideal N in a partially ordered vector space X is a subset of the positive cone that is closed under addition and multiplication by positive scalars and is such that if x is in N and y is in X with $0 \leq y \leq x$, then y is in N.

In §2 we show that in a C^*-algebra the positive parts of norm-closed left ideals are just the norm-closed order ideals, and that the norm-closure of an order ideal is again an order ideal. The latter fact is non-trivial, as it does not appear to follow from purely geometrical considerations. Analogous results are then proved for von Neumann algebras relative to the ultra-weak topology. Two-sided ideals are also discussed.

§3 is devoted to a new proof of Tomita's polar decomposition in the more general setting of the pre-dual of a von Neumann algebra. Several technical results needed in §4 and §6 are included.

In §4 we turn to order ideals in the pre-duals of von Neumann algebras and the duals of C^*-algebras. For the latter, two topologies are relevant: the norm and

Received July 23, 1962. This work has been partially supported by the National Science Foundation under Grant No. G19022.