FUNCTIONS WITH POSITIVE REAL PART IN A HALF-PLANE

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This paper attempts a simple, unified approach to the proofs of several theorems in the theory of "positive" functions. Some of these results are new and some are generalizations of known theorems. Several simple proofs of standard theorems on positive harmonic functions are included as applications of the theory.

Let $z = x + iy = re^{i\theta}$, $f = u + iv$ be analytic and single-valued in the open right half-plane (hereafter, ORHP). We write f is positive or $f \varepsilon P$ if

$$\text{(A)} \qquad u > 0 \quad \text{for} \quad x > 0.$$

If $f \varepsilon P$ and

$$\text{(B)} \qquad f(\bar{z}) = \overline{f(z)} \quad \text{for} \quad x > 0$$

where $\bar{z} = x - iy$, then we write $f \varepsilon PR$ or f is positive real.

We note that $h_1(z) = az + ib$ and $h_2(z) = ib + a(z - ic)^{-1}$ ($a > 0$, b and c real) are positive functions that include all the linear transformations of ORHP onto itself. Moreover, if $h_1 \varepsilon PR$, then $b = 0$; if $h_2 \varepsilon PR$, then $b = c = 0$. We write $h_1[z; z_0, w_0] = w_0 + s_0(z - z_0)/x_0$ and $h_2[z; z_0, w_0] = w_0 - s_0(z - z_0)(z - z_0 + x_0)^{-1}$, where $w_0 = s_0 + it_0$ and $z_0 = x_0 + iy_0$. If we set $h_1(z_0) = w_0$, then it is easily seen that $h_1(z)$ may be written as $h_1[z; z_0, w_0]$. Similarly, if we set $h_2(z_0) = w_0$ and require that $b = t_0$, then $h_2(z)$ reduces to $h_2[z; z_0, w_0]$. It should be noted that $h_2[z; z_0, w_0]$ is a particular normalization of $h_2(z)$ such that $h_2(\infty) = it_0$. This forces $y_0 = c$ so that for any given $h_2(z)$, z_0 is not arbitrary.

Positive functions have two elementary properties which we shall use repeatedly and which we state without proof.

(i) *If f and g are positive, then $f + g$, $1/f$ and $af + ib$ ($a > 0$, b real) are positive.*

(ii) *If $f \varepsilon P$ and $z = iy$ (y real) is a point of analyticity of f at which f vanishes, then $f'(iy) > 0$.*

Property (ii) is usually proved for positive real f. However, the restriction that $f(\bar{z}) = (f(z))^-$ is superfluous and the theorem may be stated as in (ii).

THEOREM 1. *If $f \varepsilon P$ and f is not a linear transformation of* ORHP *onto itself, then for every z_0 in* ORHP, *$f^* \varepsilon P$, where*

$$(1) \qquad f^*(z) = h_2[z; z_0, 1] \frac{h_1[z; z_0, f(z_0)] - f(z)}{f(z) - h_2[z; z_0, f(z_0)]}.$$

Received April 7, 1961.