THE EQUATION $\bar{t}at = b$ IN A COMPOSITION ALGEBRA

By Barth Pollak

Introduction. In a previous paper [3], I studied the solution of the equation $\bar{t}at = b$ with Nt preassigned over a quaternion algebra. It turns out that the results are valid over any composition algebra so that the associativity of the quaternion multiplication is a luxury that may be eliminated.

1. **The composition algebra C.** In this section we follow Jacobson [1] which the reader should consult for details.

Let k be a field of characteristic $\neq 2$. By a composition algebra we mean a pair (C, N) consisting of a (non-associative) algebra C over k and a quadratic form N on C to k satisfying $N(xy) = N(x)N(y)$ for all $x, y \in C$. One also makes two additional assumptions: (i) C has an identity element 1 and (ii) the bilinear form $\frac{1}{2}[N(x + y) - N(x) - N(y)]$ is non-degenerate. One then shows that any composition algebra is an alternative algebra with involution $x \to \bar{x}$ such that $x\bar{x} = N(x)1$ and $x + \bar{x} = S(x)1$ with $N(x)$ and $S(x)$ in k. $N(x)$ and $S(x)$ are called respectively the *norm* and *trace* of x, and one has $x^2 - S(x)x + N(x)1 = 0$.

It is an important result (originally investigated by Hurwitz) that dim $C = 1$, 2, 4 or 8. In dim 2 we have a (commutative) quadratic algebra, in dim 4 a (generalized) quaternion algebra and in dim 8 a (generalized) Cayley-Dickson algebra. For the problem discussed in this paper dim 2 is trivial and dim 4 has already been studied in [3] so that our attention will be focused on dim 8. In most cases we will be able to cut down to a quaternion subalgebra and use the results of [3]. However this will not always be possible and thus in §§5 and 6 a more delicate analysis is necessary.

2. **Statement of the problem.** Let $a, b \in C$. Let $\sigma \in k^*$ (the multiplicative group of non-zero elements of k). We ask: does there exist $t \in C$ such that $\bar{t}at = b$ with $Nt = \sigma$? If such a t exists, we must have $Nb = \sigma^2 Na$ and $Sb = \sigma Sa$. Writing $a = Sa/2 + a_1$ and $b = Sb/2 + b_1$ we see that $\bar{t}at = b$ with $Nt = \sigma$ if and only if $\bar{t}a_1t = b_1$ with $Nt = \sigma$. Hence no generality is lost in assuming $Sa = Sb = 0$, that is, that a and b are *pure*. Thus we state our problem in the following form:

Suppose $a, b \in C$ and are pure. Suppose $\sigma \in k^*$. Finally, let $Nb = \sigma^2 Na$. Give necessary and sufficient conditions for the existence of t in C with

$$(H) \qquad \bar{t}at = b \quad \text{and} \quad Nt = \sigma.$$

In the remainder of this paper, a and b will denote pure, non-zero elements of C such that $Nb = \sigma^2 Na$ for some $\sigma \in k^*$. Finally we will assume in the rest of this paper with the exception of §7 that dim $C = 8$.

Received February 27, 1961; in revised form, January 15, 1962.