HOMOGENEOUS AND PERIODIC SPACES OF ENTIRE FUNCTIONS

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Recent work with entire functions has led to the concept of a Hilbert space whose elements are entire functions and which has these three properties:

(H1) Whenever $F(z)$ is in the space and has a non-real zero w, the function $F(z)(z - \bar{w})/(z - w)$ is in the space and has the same norm as $F(z)$.

(H2) For every non-real number w, the linear functional defined on the space by $F(z) \to F(w)$ is continuous.

(H3) Whenever $F(z)$ is in the space, the function $F^*(z) = \bar{F}(\bar{z})$ is in the space and has the same norm as $F(z)$.

If $E(z)$ is an entire function which satisfies

$$(1) \qquad | E(\bar{z}) | < | E(z) |$$

for $y > 0 (z = x + iy)$, let $\mathcal{H}(E)$ be the set of entire functions $F(z)$ such that

$$|| F ||^2 = \int | F(t) |^2 | E(t) |^{-2} dt < \infty$$

with integration on the real axis, and

$$| F(z) |^2 \leq || F ||^2 \frac{| E(z) |^2 - | E(\bar{z}) |^2}{2\pi i(\bar{z} - z)}$$

for all complex z. Then, $\mathcal{H}(E)$ is a Hilbert space of entire functions which satisfies H1, H2, and H3. In this case, we write $E(z) = A(z) - iB(z)$ where $A(z)$ and $B(z)$ are entire functions which are real for real z, and

$$K(w, z) = \frac{B(z)\bar{A}(w) - A(z)\bar{B}(w)}{\pi(z - \bar{w})}.$$

Then, for each complex number w, $K(w, z)$ belongs to $\mathcal{H}(E)$ as a function of z and

$$F(w) = \langle F(t), K(w, t) \rangle$$

for every $F(z)$ in $\mathcal{H}(E)$. As shown in [7], every Hilbert space of entire functions which satisfies H1, H2, and H3, and which contains a non-zero element, is equal isometrically to some such $\mathcal{H}(E)$.

General properties of such spaces have been developed [8], [9], [10], but there has been little opportunity to discuss particular examples. There are several interesting special cases which have guided us in developing the general theory, and we will now present them. To motivate the construction, consider the function $E(z) = \exp(-iz)$, so that $A(z) = \cos z$ and $B(z) = \sin z$. The space

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