THE GEOMETRY OF FUNCTIONS HOLOMORPHIC IN THE UNIT CIRCLE, OF ARBITRARILY SLOW GROWTH, WHICH TEND TO INFINITY ON A SEQUENCE OF CURVES APPROACHING THE CIRCUMFERENCE

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1. **Introduction.** Let $\mu(r)$ be a given *positive increasing* function on [0, 1) with $\lim_{r\uparrow 1} \mu(r) = \infty$. It is well known that there exists a function $H(\zeta)$, holomorphic in $|\zeta| < 1$, whose maximum modulus $M(r)$ satisfies

$$M(r) \leq \mu(r) \tag{1}$$

and is such that for an appropriate sequence $s_n \uparrow 1$

$$\min_{|\zeta| = s_n} |H(\zeta)| \rightarrow \infty, \qquad n \rightarrow \infty. \tag{2}$$

Such functions may be constructed by the use of gap series, Lusin and Priwaloff [5]; MacLane [7], or via an infinite product, Bagemihl, Erdös, and Seidel [1]. The object, Theorem 2, of the present note is to construct such a function geometrically by starting with the Riemann surface $\mathcal{S}$ onto which $w = H(\zeta)$ maps $|\zeta| < 1$. The surface $\mathcal{S}$ is fairly simple and makes such functions $H(\zeta)$ seem less outlandish. The essence of the argument is in showing that $\mathcal{S}$ is hyperbolic and that (1) is satisfied. In place of (2) we shall obtain the following: There exists an expanding set of Jordan curves $C_n \subset \{|\zeta| < 1\}$, $n \geq 1$, where C_1 contains $\zeta = 0$ in its interior and C_{n+1} contains C_n in its interior, such that

$$|H(\zeta)| = \rho_n, \qquad \zeta \in C_n, \tag{2a}$$

where $\{\rho_n\}_1^\infty$ is a sequence of constants satisfying

$$0 < \rho_n \uparrow \infty. \tag{3}$$

Now if this is the case, then for any $\epsilon > 0$ and $n > n_0(\epsilon)$, C_n must be contained in the annulus

$$1 - \epsilon < |\zeta| < 1$$

and separate $|\zeta| = 1 - \epsilon$ and $|\zeta| = 1$. If D_n is the interior of C_n and $\mathcal{S}_n$ is the corresponding part of $\mathcal{S}$, then we observe that $\mathcal{S}$ is exhausted by a sequence of components, $\mathcal{S}_n$, where $\mathcal{S}_n$ is a finite-sheeted unbordered covering of $|w| < \rho_n$. The method of construction is to commence with a whole class of such surfaces $\mathcal{S}$ and pick out one that is hyperbolic and such that (1) is satisfied. The principal tool used is the kernel theory of Carathéodory [3]; a more general exposition of kernels may be found in [6].

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