# ABSOLUTELY CONTINUOUS OPERATORS

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1. **Introduction.** In this paper we continue the study of general spectral theory for operators in Hilbert space, begun in [6], [7] (see also [5]). This work depends, essentially at several points, on the use of unitary dilations of contractions. The unitary dilation $U$ of a contraction $A$ acting on a Hilbert space $H$ is an essentially unique unitary operator acting on a larger Hilbert space $K$ such that $A^n x = PU^n x$, $A^{*n}x = PU^{-n}x$ for all $n$ and $x \varepsilon H$, where $P : K \to H$ is the projection of $K$ onto $H$ (see [6], [4] for the basic details). In the papers [7], [5] results on functional calculus are expounded. The main theorem of this paper is an extension and improvement of these results, for a restricted class of operators, which we term absolutely continuous. These are contractions whose positive-operator-valued resolutions (strong operator measures [6]) are absolutely continuous, in a natural sense, with respect to Lebesgue measure on the unit circle. The strong operator measure $F$ of a contraction $A$ may be described as follows. If $U = \int z\, dE(z)$ is the unitary dilation, with spectral measure $E$, of $A$, then $F(\sigma) = PE(\sigma)$ for Borel sets $\sigma$. We say $A$ is absolutely continuous if $(F(\cdot)x, y)$ is absolutely continuous with respect to Lebesgue measure, for every pair $x, y \varepsilon H$. For such operators $A$ the main theorem says that $f(A)$ is defined for $f \varepsilon L_\infty(0, 2\pi)$, that for $f$ in the class $H^\infty$ of functions bounded and analytic in $\{| z | < 1\}$ the map $f \to f(A)$ is a homomorphism, and that if the spectrum of $A$ is sufficiently full, the map is an isometry. The isometry, which is the main new feature, holds also for functions of $A^*$ when it does for those of $A$. We have two applications, to one-parameter semi-groups and to Toeplitz matrices, and we discuss the continuity of $f(A)$ as a function of $A$, extending a lemma of Kaplansky [2]. As a corollary of this discussion we have the interesting result that, with certain limitations, the strong operator measure of a contraction is, on each fixed Borel set, a continuous function of the contraction. Finally, we prove that an absolutely continuous operator has a bounded "derivative" if and only if its spectral radius is (strictly) less than 1. We do not have a characterization of absolutely continuous operators, except in this case, and the several examples given indicate that they exist in variety.

2. **Absolutely continuous operators.** Let $A$ be a contraction ($\| A \| \leq 1$) on Hilbert space $H$, with strong operator measure $F$, so that $A = \int_0^{2\pi} e^{it}\, dF(t)$ (see [4] or [6]). We write $\| f \|_{\infty F} = \inf \{\sigma \mid F(\Lambda - \sigma) = 0\} \sup \{| f(z) | \mid z \varepsilon \sigma\} = \text{Ess sup} \{| f(z) | \mid z \varepsilon \Lambda(F)\}$, where $\Lambda$ is the support of $F$, a subset of the unit circle $C$, and we write $L_{\infty F}(\Lambda) = \{f \mid \| f \|_{\infty F} < \infty\}$ with the usual identification. We

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