COMPACTNESS IN FUNCTION SPACES

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Compactness in the function space topology of almost uniform convergence is the concern of this paper [2]. As seen in the space of compact or weakly compact operators [4], there is a need for the knowledge of this compactness. Applications to compactness in other topologies, such as the strong operator topology, are immediate (see §5). This paper is also an attempt to unify and clarify some aspects of compactness in function spaces.

In order to develop needed tools, §1 contains a modernization of the repeated limit. Theorem 1.6 is a basic theorem for changing the order of iteration in an iterated limit. The repeated limit is utilized in §2 to characterize convergent nets, a cluster point of a net, and Cauchy nets in the topology of almost uniform convergence.

The definition of almost compact subsets of a function space is given in §3 for the purpose of investigating compact sets without being unduly influenced by either the structure of the compact subsets of the range space or the "size" of the function space. Theorem 3.3, which appears to be one of the most general theorems of the Eberlein type [5], gives the equivalence of almost compact and almost countably compact sets. The section concludes with a needed result on totally bounded sets for the topology of almost uniform convergence.

The inherent duality of almost compactness is displayed in Theorem 4.3 which contains nine equivalent statements and two additional ones which conditionally join the equivalence. Each of the many statements, sometimes with additional restrictions, has in essence been used by other authors to imply a type of compactness. Statements (ii), (iii) and (xi) were suggested by an investigation due to J. L. Kelley. Statements (iii), (vii), (vii′) and (ix) are modifications of conditions appearing in a paper by R. G. Bartle [1], and statements (vii), (viii), (vi) and (xi) are modifications of conditions used by A. Grothendieck [6]. The present formulation explicitly displays their relationship and extends their usefulness.

Applications to the topology of pointwise convergence on a space of continuous functions are given in §5. Theorems 5.5 and 5.6 are examples of the numerous results that are immediately available. The section terminates with an application to the theory of weak pairings [1].

1. **The repeated limit.** Consider a function $\phi(\alpha, \beta)$ defined whenever α belongs to a directed set A and β belongs to a directed set B_α from the family

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