NECESSARY CONDITION ON C-FRACTIONS OF ALGEBRAIC FUNCTIONS

By E. P. Merkes and W. T. Scott

1. **Introduction.** Liouville [1]; see also [2, vol. 1; 127] used simple continued fractions to give the first proof of the existence of transcendental numbers. His proof depends on a necessary condition for a simple continued fraction to be the expansion of a real root of an irreducible algebraic equation. In this paper a necessary condition is obtained for a C-fraction to correspond to a non-rational algebraic function. This result is, in a sense, an analog for C-fractions of Liouville's necessary condition for simple continued fractions.

Theorem. *Let* $w = f(z) = \sum_{n=1}^{\infty} c_n z^n$ *be regular in a neighborhood of* $z = 0$ *and satisfy there the irreducible algebraic equation*

$$\sum_{j=0}^{k} P_j(z) w^j = 0, \qquad k \geq 2, \tag{1.1}$$

where $P_j(z)$, $(j = 0, 1, \cdots, k)$, *is a polynomial in* z *of degree not exceeding* p *and* $P_0(0) = 0$. *Then for fixed* $p \geq 1$ *and* $k \geq 2$ *the exponents* δ_{n+1} *of the* C-*fraction expansions*

$$f(z) \sim \frac{d_1 z^{\delta_1}}{1} + \frac{d_2 z^{\delta_2}}{1} + \cdots + \frac{d_n z^{\delta_n}}{1} \cdots \tag{1.2}$$

have least upper bounds $M_n(p, k)$,

$$\delta_{n+1} \leq M_n(p, k). \tag{1.3}$$

Moreover, for $k > 2$,

$$M_0(p, k) \leq p, \qquad M_n(p, k) \leq k(k-1)^{n-1} p, \qquad (n = 1, 2, \cdots), \tag{1.4}$$

and for $k = 2$,

$$M_0(p, 2) \leq p, \qquad M_{2n-1}(p, 2) \leq 2^n(p-1) + 2 \tag{1.5}$$

$$M_{2n}(p, 2) \leq 2^n(p-1) + 2, \qquad (n = 1, 2, \cdots).$$

The proof is contained in the following sections.

2. **A transformation.** For a non-negative integer n let $w_n = f_n(z) = \sum_{j=1}^{\infty} c_j^{(n)} z^j$ be a regular solution of the irreducible algebraic equation

$$G_n(w, z) \equiv \sum_{j=0}^{k} P_j^{(n)}(z) w^j = 0, \qquad (P_0^{(n)}(0) = 0, k \geq 2), \tag{2.1}$$

Received August 15, 1960.