SOME EXTREMAL THEOREMS FOR MULTIVALENTLY STAR-LIKE FUNCTIONS

By John Bender

1. **Introduction.** The concept of a generalized star-like function has been used previously by several authors.

Definition (1.1). *A function $f(z)$ is said to be a member of the class $\mathcal{S}(p, q)$, where p and q are positive integers with $p \geq q$, if and only if there is a positive number ρ such that*

$$\mathcal{R}\left[\frac{zf'(z)}{f(z)}\right] > 0, \qquad \rho < |z| < 1, \tag{1.1}$$

$$\int_0^{2\pi} \mathcal{R}\left[\frac{zf'(z)}{f(z)}\right] d\theta = 2\pi p, \qquad z = re^{i\theta}, \qquad \rho < r < 1, \tag{1.2}$$

and

$$f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n, \qquad |z| < 1. \tag{1.3}$$

We shall also say that $f(z)$ is multivalently star-like of order (p, q) with respect to the origin and in the unit circle $|z| < 1$.

$\mathcal{S}_1(p, q)$ shall denote the class of multivalently star-like functions of class $\mathcal{S}(p, q)$ which satisfy (1.1), (1.2), and (1.3) on the circle $|z| = 1$.

In §2, we shall display a representation for a function $f(z)$ of class $\mathcal{S}(p, q)$ in terms of a function $\phi(z)$ of class $\mathcal{S}$ (1, 1) and the zeros of $f(z)$.

Definition (1.2). *A function $f(z)$ is said to be a member of the class $\mathcal{F}(p, q)$ if and only if $f(z)$ has the representation given by equation* (2.1).

We shall prove that the class $\mathcal{S}(p, q)$ is a proper sub-class of $\mathcal{F}(p, q)$ when $1 \leq q < p$ and that $\mathcal{S}(p, p)$ and $\mathcal{F}(p, p)$ are equivalent for all positive integers p. Moreover, each member of $\mathcal{F}(p, q)$ is shown to be p-valent and a limit of a sequence of functions belonging to $\mathcal{S}(p, q)$.

The representation theorem permits us to obtain elementary proofs of the sharp bounds for $|f(z)|$, $|f'(z)|$, and $|a_n|$ which were previously proved by Goodman [1]. In addition, we have obtained sharp bounds from below for $|f'(z)|$ provided that $|z|$ is sufficiently small and a sharp bound below is obtained for $\mathcal{R}[zf'(z)/f(z)]$ when $|z|$ not exceed the minimum of the absolute values of the zeros of $f(z)$.

We remark that we shall use the symbol $K(\alpha, \beta, f, \cdots)$ to denote a positive constant depending on $\alpha, \beta, f, \cdots$.

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