TRACE PROPERTIES OF SEMIGROUPS OF MATRICES WITH QUATERNION ELEMENTS

By N. A. Wiegmann

1. **Introduction.** In the theory of group representations involving matrices with complex elements, the trace of a matrix is of great importance. When the elements of a matrix do not lie in a commutative domain (see [1], for example), the trace ceases to have much utility, in general. Here matrices with real quaternion elements are considered, and it is seen that in this instance the trace does have a corresponding value. See [2] in this connection.

In the following a lemma is first obtained, and thereafter analogs of several theorems which are standard for the complex case are developed.

2. **Irreducible semigroups.** Every $n \times n$ matrix with real quaternion elements can be written in the form $A = A_1 + jA_2$ where A_1 and A_2 are complex matrices. To each A there corresponds a $2n \times 2n$ complex matrix:

$$\text{(i)} \qquad A^* = \begin{bmatrix} A_1 & -A_2^C \\ A_2 & A_1^C \end{bmatrix}$$

where A_1^C denotes the complex conjugate of the matrix A_1 and this correspondence is an isomorphism up to a certain point (see [3]). If $\mathfrak{A}$ denotes a set of $n \times n$ matrices with quaternion elements which forms a semigroup, finite or infinite, under matrix multiplication, and if $\mathfrak{A}^*$ denotes the set of $2n \times 2n$ complex matrices obtained from $\mathfrak{A}$ as above, then $\mathfrak{A}^*$ is a semigroup of complex matrices under matrix multiplication.

It may happen that for a given $\mathfrak{A}$ there exists a nonsingular quaternion matrix P such that $P\mathfrak{A}P^{-1} = \mathfrak{C}$ is a complex set or there may exist no such P. (See [4; 340].)

Lemma. *Let $\mathfrak{A}$ be a semigroup of $n \times n$ quaternion matrices which is (quaternion) irreducible and which is not similar to a complex set. Then the complex semigroup $\mathfrak{A}^*$ is complex irreducible.*

From the given conditions by [4, Theorem 3] $\mathfrak{A}$ has l-rank n^2 (where the l-rank is the maximum number of left linearly independent matrices in $\mathfrak{A}$). By the lemma following Theorem 3 [4; 341], the number of right linearly independent "K-matrices" is $n^2 - n^2 = 0$, where $K = (k_{\lambda\kappa})$ is a "K-matrix" relative to $\mathfrak{A}$ if the trace of $AK = \chi(AK) = \sum_{\kappa,\lambda=1}^{n} a_{\kappa\lambda}k_{\lambda\kappa} = 0$ for each A in $\mathfrak{A}$. If one considers the complex semigroup $\mathfrak{A}^*$, this means there exists no nonzero $2n \times 2n$ complex matrix K^* in the $*$-form (i) such that $\chi(A^*K^*) = 0$ for all A^* in $\mathfrak{A}^*$.

Received March 14, 1961.