GENERATING FUNCTIONS FOR POWERS OF FIBONACCI NUMBERS

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1. **Introduction.** The Fibonacci numbers, f_n, may be defined by $f_0 = f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$, $n = 2, 3, \cdots$. Their generating function is

$$f_1(x) = \sum_{n=0}^{\infty} f_n x^n = (1 - x - x^2)^{-1}.$$

The similar generating function for the k-th power of f_n is defined as

$$f_k(x) = \sum_{n=0}^{\infty} f_n^k x^n.$$

S. W. Golomb [2] (see also [1, 194, Example 32]) has found effectively that

$$(1 - 2x - 2x^2 + x^3)f_2(x) = 1 - x$$

and thus raised the question of the character of similar expressions for $k = 3, 4, \cdots$. A short proof of this expression is as follows. First, it is familiar that

$$f_n^2 - f_{n-1}f_{n+1} = (-1)^n, \qquad n = 0, 1, 2, \cdots.$$

Then

$$\begin{aligned} f_n^2 &= (f_{n-1} + f_{n-2})^2 \\ &= f_{n-1}^2 + f_{n-2}^2 + 2(f_n - f_{n-2})f_{n-2} \\ &= f_{n-1}^2 - f_{n-2}^2 + 2(f_{n-1}^2 + (-1)^n) \end{aligned}$$

or

$$f_n^2 - 3f_{n-1}^2 + f_{n-2}^2 = 2(-1)^n, \qquad n = 1, 2, \cdots$$

while $f_0^2 = 1$. Hence

$$(1 - 3x + x^2)f_2(x) = 1 - 2xf_0(-x)$$

where $(1 - x)\, f_0(x) = 1$. Simplifying leads to Golomb's result.

In similar fashion it is found that

$$f_n^3 - 4f_{n-1}^3 - f_{n-2}^3 = 3(-1)^n f_{n-1}, \qquad n = 1, 2, \cdots$$

$$f_n^4 - 7f_{n-1}^4 + f_{n-2}^4 = 8(-1)^n f_{n-1}^2 + 2, \qquad n = 1, 2, \cdots$$

which correspond to the generating function relations

$$\begin{aligned} (1 - 4x - x^2)f_3(x) &= 1 - 3xf_1(-x) \\ (1 - 7x + x^2)f_4(x) &= 1 - 8xf_2(-x) + 2xf_0(x). \end{aligned}$$

Received May 5, 1961.