THE SEPARATION OF S^3 BY A DOUBLE TORUS

By C. H. Edwards, Jr.

Alexander [1] proved that, if T is a polyhedral torus (closed orientable surface of genus 1) in the 3-sphere S^3, then the closure of at least one of the components of $S^3 - T$ is a solid torus of genus 1 (the topological product of a disk and a circle). Simple examples show that additional conditions must be imposed upon the imbedding in S^3 of a polyhedral double torus (closed orientable surface of genus 2) in order to insure that the closure of at least one of the components of its complement will be a solid torus of genus 2. It is the purpose of this paper to provide such conditions, framed in the spirit of the following well-known theorem: If T is a polyhedral torus in S^3, with A a component of $S^3 - T$, and if m is a non-nullhomologous curve on T bounding a disk D such that $\text{Int } D \subset A$, then $\text{Cl } A$ is a solid torus of genus 1 [2].

Suppose that T is a double torus. By an *equator* of T is meant a simple closed curve e on T such that $T - e$ consists of two components, the closure of each of which is a torus with one hole. If e is an equator on T, it is clear that e is nullhomologous on T but not nullhomotopic on T. By a *solid torus of genus* 2 is meant the union B of two solid tori of genus 1, whose intersection is a disk common to their boundaries. A solid torus B of genus 2 in S^3 is said to be *unknotted* if and only if B can be expressed as the union of two unknotted solid tori of genus 1, meeting in a disk common to their boundaries. If c is a simple closed curve on a closed surface M in S^3, and A is a component of $S^3 - M$, it will be said that *c bounds in A* if and only if there exists a disk D such that $\text{Bd } D = c$ and $\text{Int } D \subset A$. All sets considered in this paper will be implicitly assumed to be polyhedral.

Theorem 1. *Let T be a double torus in S^3. Suppose that e is an equator of T, and that m_1 and m_2 are non-nullhomologous simple closed curves on T which are separated on T by e. If A is a component of $S^3 - T$ such that e, m_1, m_2 all bound in A, then* $\text{Cl } A$ *is a solid torus of genus* 2.

Proof. Let E, M_1, M_2 be disks bounded by e, m_1, m_2 respectively, the interior of each being contained in A. Denote by C_1 and C_2 the two components of $T - e$ containing m_1 and m_2 respectively. Then clearly $T_1 = C_1 \cup E$ and $T_2 = C_2 \cup E$ are tori. Since $C_2 \cap T_1 = \square$, the connected set C_2 must be contained entirely in one component of $S^3 - T_1$. If B_1 is the other component of $S^3 - T_1$, then $B_1 \cap T = \square$. Since $\text{Int } E \subset A$ and since $\text{Int } E \subset \text{Bd } B_1$, it is clear that $A \cap B_1 \neq \square$. It follows that $B_1 \subset A$. Similarly A contains

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