# FACTORIZATION AND LATTICE THEOREMS FOR A BOUNDED PRINCIPAL IDEAL DOMAIN

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0. **Introduction.** A right ideal $aA$ of a principal ideal domain $A$ is called bounded if it contains an ideal $\neq 0$. (Ideal shall mean two-sided ideal.) The join of all ideals contained in $aA$ is then an ideal called the *bound* $a^*A = Aa^*$ of $aA$. If every right ideal of a principal ideal domain $R$ is bounded, we say that $R$ is a *bounded principal ideal domain. Henceforth $R$ shall always denote a bounded principal ideal domain and $aR$ shall denote a right ideal not* 0 *or* $R$. For an example of a ring $R$ see Theorem 15 of [5; 41]. *The definitions of* [3] *will be used in this paper without reference.*

The most important results given in this paper are found in §6 which we shall now explain. In a commutative principal ideal domain $S$ if $aS = p_1^{e_1}S \cap \cdots \cap p_n^{e_n}S$ where the ideals $p_i^{e_i}S$ are $\cap$ irreducible and the $p$'s are irreducible and if $a = q_1^{e_1}q_2^{e_2} \cdots q_n^{e_n}$ where the $q$'s are irreducible, then the $p$'s and $q$'s may be paired as associates, i.e., $q_i = u_i p_i$ for $u_i$ a unit. We shall show that the same type result holds in $R$. Of course, which is often the case, the associate relation in the commutative case becomes the similar relation of Ore in $R$. (See [5; 33]). Let $aR = b_1R \cap \cdots \cap b_sR$ where the $b_iR$ have distinct tertiary radicals (defined later) and let $a = c_1c_2 \cdots c_k$ where the $c$'s are irreducible, then if $b_i = d_{i1} \cdots d_{in_i}$ where the $d$'s are irreducible, we prove that the $d$'s and $c$'s can be paired into similar pairs. This result points out a relationship between the lattice structure of $R$ and the factorization into irreducible elements in $R$. In addition, in §6 we show that if $a$ is $\cap$ irreducible and $a = c_1 \cdots c_n = b_1 \cdots b_n$ where the $b$'s and $c$'s are irreducible, then $c_iR = b_iR$ for $i = 1, 2, \cdots, n-1$ and $Rb_i = Rc_i$ for $i = 2, 3, \cdots, n$. This factorization is much stronger than given by Theorem 5 of [5; 34].

In the first five sections we shall discuss properties of $\cap$ irreducible right ideals of $R$ and uniqueness theorems for the representation of an arbitrary right ideal as the intersection of $\cap$ irreducible right ideals. The result will follow mainly from three sources, namely: Chapter 3 of Jacobson's Theory of Rings [5], a paper by L. Lesieur and R. Croisot [6], and the author's publications [2] and [3]. Certain relationships between the different radicals defined in these publications are proved here.

1. **Intersection irreducible right ideals of R.** If $a \in R$ is neither 0 or a unit and $R - aR$ is indecomposable, then $a$ is called *indecomposable.* Theorem 24 of [5; 46] tells us that if $q$ is indecomposable, then $R - qR$ has only one composition series. Then Theorem 1.1 of [3] implies that $q$ is right irreducible since

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