ORTHOGONAL SAW-TOOTH FUNCTIONS

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Abstract

Harrington and Cell have recently defined an interesting set of square-wave functions. They have studied the properties of this set of functions and have shown them to be orthogonal and complete in L_2. This note carries out a similar study but starts with a saw-tooth wave rather than a square wave.

1. **An operational method.** This note considers the sequence of functions $r(x), r(2x), r(3x), \cdots$. Here $r(x)$ is a function of period 1 with

$$r(x) = \tfrac{1}{2} - x, \qquad 0 < x < 1; \qquad r(0) = 0. \tag{1}$$

Thus $\frac{1}{2} - r$ is the fractional part of x. The Schmidt orthogonalization process is applied to this sequence, and a new sequence $r_1(x), r_2(x), r_3(x), \cdots$ results which is orthogonal on the interval $0 \leq x \leq 1$. The solution is given by the formula

$$r_n(x) = \sum_{k \mid n} \frac{\mu_k}{k} r\left(\frac{n}{k} x\right) \tag{2}$$

where μ_k is the Möbius function. Moreover the norm of the function $r_n(x)$ is given by

$$12 \int_0^1 r_n^2 \, dx = \sum_{k \mid n} \frac{\mu_k}{k^2}. \tag{3}$$

The problem treated in this note was suggested by the work of Harrington and Cell on square-wave functions [1]. The writer is indebted to these authors for seeing a preliminary account of their theory.

To carry out a proof of (2) and (3) it is first noted that the function $r(x)$ has the Fourier series

$$r(x) = \sum_1^\infty \frac{\sin 2\pi n x}{\pi n}. \tag{4}$$

Then the orthogonality of the terms of (4) gives

$$\int_0^1 r(Mx) r(Nx) \, dx = \sum_1^\infty \sum_1^\infty \frac{\delta(mM, nN)}{2\pi^2 mn}. \tag{5}$$

Here M and N are arbitrary integers and $\delta(x, y)$ denotes the Kronecker function. Thus $\delta = 0$ unless $m = Nk/h$ and $n = Mk/h$ where $h = h(M, N)$ is the greatest

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