A BILINEAR GENERATING FUNCTION FOR THE HERMITE POLYNOMIALS

BY L. CARLITZ

1. The bilinear generating function of Mehler [2; 194]

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\,\frac{t^n}{2^n n!} = (1-t^2)^{-\frac{1}{2}} \exp \frac{2xyt-(x^2+y^2)t^2}{1-t^2}, \tag{1}$$

where

$$H_n(x) = \sum_{2r\le n} (-1)^r \frac{n!\,(2x)^{n-2r}}{r!\,(n-2r)!}, \tag{2}$$

is well known. In the present note we examine the sum

$$\Phi(t) = \sum_{n=0}^{\infty} H_n(x)H_n(y)\,\frac{t^n}{(n!)^2}. \tag{3}$$

Making use of (2), it is evident that

$$\begin{aligned}\Phi(t) &= \sum_{n=0}^{\infty} t^n \sum_{2r\le n} (-1)^r \frac{(2x)^{n-2r}}{r!\,(n-2r)!} \sum_{2s\le n} (-1)^s \frac{(2y)^{n-2s}}{s!\,(n-2s)!} \\ &= \sum_{r,s=0}^{\infty} \frac{(-1)^{r+s}}{r!\,s!} \sum_{n\ge \max(2r,2s)} \frac{(2x)^{n-2r}(2y)^{n-2s}t^n}{(n-2r)!\,(n-2s)!}.\end{aligned} \tag{4}$$

For $r \ge s$ the inner sum on the extreme right is equal to

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(2x)^n(2y)^{n+2r-2s}t^{n+2r}}{n!\,(n+2r-2s)!} \\ &\qquad = \left(\frac{y}{x}\right)^{r-s} t^{r+\epsilon} \sum_{n=0}^{\infty} \frac{(2x)^{n+r-s}(2y)^{n+r-s}t^{n+r-s}}{n!\,(n+2r-2s)!} = \left(\frac{y}{x}\right)^{r-s} t^{r+s} I_{2r-2s}(4\sqrt{xyt}),\end{aligned}$$

in the usual notation for Bessel functions of purely imaginary argument.

For $r \le s$ we find similarly that the inner sum is equal to

$$\left(\frac{x}{y}\right)^{s-r} t^{s+r} I_{2s-2r}(4\sqrt{xyt}).$$

Thus (4) becomes

$$\Phi(t) = \sum_{r\ge s} \frac{(-1)^{r+s}}{r!\,s!} \left(\frac{y}{x}\right)^{r-s} t^{r+s} I_{2r-2s}(4\sqrt{xyt})$$

Received March 24, 1961. Supported in part by National Science Foundation grant G 9425.