# DENSITY TOPOLOGY AND APPROXIMATE CONTINUITY

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It was observed in [1] that the approximately continuous functions on Euclidean $n$ space, $E_n$, are continuous if the space is given the appropriate topology. A set $S$ is open in this topology, and is called $d$-open, if it is measurable and if the metric density of $S$ exists and is equal to 1 at every point of $S$. The topology is called the $d$-topology.

In this paper, we ask whether or not the $d$-topology is the coarsest one for which the approximately continuous functions are continuous. This is the same as asking whether or not $E_n$, with the $d$-topology, is a completely regular space. For $E_1$, the answer is yes, and is obtained as a consequence of the known fact (Lusin-Menchoff Theorem [5]) that if $E$ is a Borel set, $X \subset E$ is closed, and $E$ has metric density 1 at every $x \in X$, then there is a perfect set $P$ such that $X \subset P \subset E$ and $P$ has metric density 1 at every $x \in X$.

For $n > 1$, a distinction must be made between ordinary and strong metric density. For the case of ordinary metric density, an analogue of the Lusin-Menchoff Theorem holds and, indeed, the associated $d$-topology, now called the $d_o$-topology, is completely regular. Surprisingly, however, the Lusin-Menchoff Theorem fails to hold for strong metric density and, moreover, the corresponding $d$-topology, now called the $d_s$-topology, is not completely regular. A coarser topology for which the strongly approximately continuous functions are continuous has as open sets $S$ those measurable sets for which the strong metric density of $S$ is 1 and the linear metric densities of $S$ are 1, in the directions of the coordinate axes, at every $x \in S$. It is not known whether or not $E_n$ is completely regular with this topology.

1. **Lusin-Menchoff Theorem.** For $E \subset E_n$, $x \in E_n$, $E$ measurable, we denote the upper and lower ordinary metric densities of $E$ at $x$ by $\bar{d}_o(x, E)$ and $\underline{d}_o(x, E)$, respectively. Replacing the subscript "$o$" by "$s$", we obtain the upper and lower strong metric densities of $E$ at $x$. If $\bar{d}_o(x, E) = \underline{d}e(x, E)$, we denote the common value by $d_o(x, E)$, and call it the ordinary metric density of $E$ at $x$. We adopt an analogous convention for strong metric densities (see [4; 106, 128] for the definitions). If there is no reason to distinguish, we use $d(x, E)$ for the ordinary or strong metric density. In this paper, *interval* means *oriented interval*, $| A |$ denotes the Lebesgue measure of $A$, and $\delta(A)$ stands for the diameter of $A$.

Lemma 1. *Let $B \subset E_n$ be a Borel set, and let $x \in B$ with $d(x, B) = 1$. There is a perfect set $K$ with $x \in K$ and $K \subset B$.*

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