CAYLEY DIAGRAMS OF BINARY SYSTEMS

By R. Artzy

1. **Introduction.** The theory and applications of Cayley diagrams of groups (also called "color groups", "Dehnsche Gruppenbilder") have been studied thoroughly (cf. [4]). We propose here the application of Cayley diagrams to more general binary systems. It will turn out that some of the most beautiful properties of Cayley diagrams for groups are not applicable to the generalizations. On the other hand, several applications, in particular those to principal isotopes of loops with identities, appear to justify the generalization.

The definitions and terminology for graphs and for binary systems are in accordance with the usage in [1] and [2], respectively.

2. **Basic concepts.** (i) We consider a graph Γ whose set of vertices is V and whose set or arcs (oriented edges) is E, satisfying the requirement that there exist a single-valued mapping α of E into V such that no two arcs starting from the same vertex have the same image under α. If $e \in E$, $v \in V$, $(e)\alpha = v$, we shall say: *e has the color v*. We call Γ the *Cayley diagram* of a halfgroupoid $(G, \cdot)$ if there is a biunique mapping β of G onto V, such that for all g_1, g_2, g_3 in G, satisfying $g_1 g_2 = g_3$, one of the arcs $(g_2)\beta\alpha^{-1}$ exists and leads from $(g_1)\beta$ to $(g_3)\beta$. That this arc is unique follows from the above requirement for α. Using the color notation we can express this by saying that exactly one arc of color g_2 leads from the vertex corresponding to g_1 to the vertex corresponding to g_3. (Note that "loops", i.e. arcs leading from a vertex to itself, and multiple arcs of different colors connecting two vertices are not excluded.)

The following statements (ii) to (v) can now be easily verified.

(ii) The halfgroupoid is *cancellative* if and only if its Cayley diagram satisfies the two requirements: (a) No two distinct arcs lead from one vertex to another, (b) No two arcs of the same color terminate at the same vertex. The validity of (a) makes possible the following notation: If an arc of color x leads from a vertex A to a vertex B, we shall write col $AB = x$.

(iii) The graph Γ is the Cayley diagram of a *groupoid* if and only if arcs of all colors of V originate from each of the vertices.

(iv) Γ belongs to a *quasigroup* if and only if, in addition to (iii), exactly one arc leads from any given vertex to any other vertex, and if exactly one arc of each of the colors of V terminates in each of the vertices.

(v) Γ represents a *loop* if and only if, in addition to (iv), all arcs (or rather edges) connecting the vertices with themselves are of the same color, say u, and if for every vertex x, col $(u, x) = x$. In the following, these self-connecting

Received March 7, 1961. This research was supported in part by the G. Swope Research Fund while the author was at the Israel Institute of Technology.