A NOTE ON THE SINGULAR VALUES OF THE PRODUCT OF TWO MATRICES

BY ALI R. AMIR-MOÉZ

In this note we answer one of the questions which has not been established in [1], that is, to estimate the real and imaginary singular values of the product of two matrices.

1. **Definitions and notations.** For an n-by-n matrix A, with real or complex elements, the eigenvalues of $(A + A^*)/2$ and $(A - A^*)/2i$ are respectively called the real and imaginary singular values of A. Here A^* is the conjugate transpose of A. It is well known that A^*A and AA^* have the same eigenvalues and these eigenvalues are non-negative. The non-negative square roots of these eigenvalues are called absolute singular values of A.

If $j_p \leq i_p$ for $p = 1, \cdots, k$, we write $(j_1, \cdots, j_k) \leq (i_1, \cdots, i_k)$. Given any sequence $i_1 \leq \cdots \leq i_k$ of integers such that $i_p \geq p$ for all p, let $(i_1', \cdots, i_k')$ denote the strictly increasing sequence of positive integers such that

(a) $(i_1', \cdots, i_k') \leq (i_1, \cdots, i_k)$

(b) $(j_1, \cdots, j_k) \leq (i_1', \cdots, i_k')$ whenever $(j_1, \cdots, j_k)$

is a strictly increasing sequence of positive integers which is $\leq (i_1, \cdots, i_k)$. It is easily seen that $(i_1', \cdots, i_k')$ is given by the formulas

$$i_k' = i_k, \quad \text{and} \quad i_p = \min(i_p, i_{p+1}' - 1) \quad \text{for} \quad p = k - 1, \cdots, 1$$

[2; 2.6].

2. THEOREM. *Let A and B be two n-by-n matrices with real or complex elements. Let $\alpha_1 \geq \cdots \geq \alpha_n$ be the absolute singular values of A and $\beta_1 \geq \cdots \geq \beta_n$ be the absolute singular values of B. Let $\lambda_1, \cdots, \lambda_n$ be the real singular values of AB such that*

$$|\lambda_1| \geq \cdots \geq |\lambda_n|.$$

Then

$$(1) \quad \tfrac{1}{2}\beta_1[\alpha_{i_1'+n-1} + \cdots + \alpha_{i_k'+n-1} + \alpha_{j_1'+n-1} + \cdots + \alpha_{j_k'+n-1}] \leq |\lambda_{(i_1+j_1-n)'}| + \cdots + |\lambda_{(i_k+j_k-n)'}|,$$

$$i_p + j_p \geq n + p, \qquad p = 1, \cdots, k, \quad \textit{and}$$

$$(2) \quad \tfrac{1}{2}\beta_n[\alpha_{i_1''-n+1} + \cdots + \alpha_{i_k''-n+1} + \alpha_{j_1''-n+1} + \cdots + \alpha_{j_k''-n+1}] \geq |\lambda_{(i_1+j_1-1)''}| + \cdots + |\lambda_{(i_k+j_k-1)''}|,$$

$$i_p + j_p \leq n - k + p + 1, \qquad p = 1, \cdots, k,$$

where, for example, the sequences $(i_1', \cdots, i_k')$ and $(i_1'', \cdots, i_k'')$ are the same as in 2.6 *and* 2.10 *of* [2]. (*In this theorem α and β may be interchanged.*)

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