FINITE GROUPS OF QUATERNION MATRICES

Dedicated to the memory of Edward Jerome Finan, late Professor of Mathematics at The Catholic University of America

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Let $\mathfrak{A}$ and $\mathfrak{B}$ be semigroups of matrices over the skew field of [real] quaternions. $\mathfrak{A}$ is said to be quaternion-similar to $\mathfrak{B}$ if and only if there exists a non-singular matrix P with quaternion coefficients such that $P^{-1}\mathfrak{A}P = \mathfrak{B}$. Otherwise $\mathfrak{A}$ is quaternion distinct from $\mathfrak{B}$. Similarly, $\mathfrak{A}$ is said to be complex-similar to $\mathfrak{B}$ if and only if there exists a non-singular matrix R with complex coefficients such that $R^{-1}\mathfrak{A}R = \mathfrak{B}$. Otherwise $\mathfrak{A}$ is complex-distinct from $\mathfrak{B}$. See [2] for further conventions. In particular, if the quaternion matrix M is written in the form $M_1 + jM_2$ where M_1 and M_2 are complex matrices, $M^* = (M_{\alpha\beta})$, $\alpha, \beta = 1, 2$ where $M_{11} = M_{22}^c = M_1$ and $M_{21} = -M_{12}^c = M_2$. (A^c indicates the complex conjugate of the matrix A.)

The purpose of this work is to determine all the quaternion-distinct, quaternion-irreducible representations of a finite group G by quaternion matrices. It would be sufficient to determine the quaternion-distinct, quaternion-irreducible constituents of the regular representation of G and then to show that every quaternion-irreducible representation of G is quaternion-similar to a constituent of the regular representation. That it is possible to do so is a consequence, in part, of the following theorem concerning the quaternion-similarity of complex-distinct semigroups.

THEOREM 1. (This theorem is similar to [1, Chapter I, §5].) *Let $\mathfrak{A}$ and $\mathfrak{B}$ be complex-distinct, complex-irreducible semigroups of complex matrices.*

(i) *$\mathfrak{A}$ is quaternion-similar to $\mathfrak{B}$ if and only if $\mathfrak{A}$ is complex-similar to $\mathfrak{B}^c$.*
(ii) *If $\mathfrak{A}$ and $\mathfrak{B}$ are quaternion-reducible, then $\mathfrak{A}$ is quaternion-distinct from $\mathfrak{B}$.*
(iii) *If $\mathfrak{A}$ is quaternion-reducible and $\mathfrak{B}$ is quaternion-irreducible, then $\mathfrak{A}$ and $\mathfrak{B}$ are quaternion-distinct.*

Proof of (i). If there is a complex matrix P such that $P^{-1}\mathfrak{A}P = \mathfrak{B}^c$, then $-jP^{-1}\mathfrak{A}Pj = -j\mathfrak{B}^c j$ or $(Pj)^{-1}\mathfrak{A}(Pj) = \mathfrak{B}$, and $\mathfrak{A}$ is quaternion similar to $\mathfrak{B}$. Conversely, if there is a quaternion matrix P such that $P^{-1}\mathfrak{A}P = \mathfrak{B}$, then $(P^{-1})^*\mathfrak{A}^*P^* = \mathfrak{B}^*$, and $\mathfrak{A} \oplus \mathfrak{A}^c$ is complex-similar to $\mathfrak{B} \oplus \mathfrak{B}^c$. Since $\mathfrak{A}$ and $\mathfrak{B}$ are complex-distinct and complex-irreducible, it follows that $\mathfrak{A}$ is complex-similar to $\mathfrak{B}^c$.

Proof of (ii). There exist quaternion matrices P and R such that $P^{-1}\mathfrak{A}P = \mathfrak{A}_1 \oplus \mathfrak{A}_1$ and $R^{-1}\mathfrak{B}R = \mathfrak{B}_1 \oplus \mathfrak{B}_1$ where $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are quaternion-irreducible semigroups [2, Theorem 1]. If $\mathfrak{A}$ is quaternion-similar to $\mathfrak{B}$, then $\mathfrak{A}_1$ is quater-

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