SOME OPERATIONAL EQUATIONS FOR SYMMETRIC POLYNOMIALS

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1. Introduction. Let $x_1, x_2, \cdots, x_k$ be k indeterminates, where k is a fixed integer > 1. For $r \geq 0$ define the linear operator

$$\Omega_r = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{k-2} & x_2^{k-2} & \cdots & x_k^{k-2} \\ x_1^r \dfrac{\partial}{\partial x_1} & x_2^r \dfrac{\partial}{\partial x_2} & \cdots & x_k^r \dfrac{\partial}{\partial x_k} \end{vmatrix}. \tag{1.1}$$

If we put

$$T = T_k = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{k-1} & x_2^{k-1} & \cdots & x_k^{k-1} \end{vmatrix}$$

and let X_j denote the cofactor of x_j^{k-1} in T, then we have

$$\Omega_r = \sum_{j=1}^{k} x_j^r X_j \frac{\partial}{\partial x_j}. \tag{1.2}$$

It is convenient also to define

$$\omega_r = T^{-1} \Omega_r . \tag{1.3}$$

If now $S = S(x_1, x_2, \cdots, x_k)$ is any symmetric polynomial, it follows that $\omega_r S$ is also symmetric; if S is homogeneous of weight N, then $\omega_r S$ is homogeneous of weight $N + r - k$. Conversely we shall show that if S is a given symmetric polynomial, then any polynomial F that satisfies the equation

$$\omega_r F = S \tag{1.4}$$

for some r is necessarily symmetric.

We next discuss the equation (1.4) for $0 \leq r \leq k$. We show that (1.4) is always solvable for values of r in this range. The case $r = k$ is particularly interesting. We find that the operator ω_k induces a non-singular linear transformation on the space R_N of symmetric polynomials of weight N. If $a_1, a_2, \cdots, a_k$ denote the elementary symmetric polynomials in $x_1, x_2, \cdots, x_k$, then it is familiar that the set of symmetric polynomials

$$a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k} \qquad (n_1 + 2n_2 + \cdots + kn_k = N) \tag{1.5}$$

Received October 13, 1960. Supported in part by National Science Foundation Grant G-9425.