A SERIES TRANSFORMATION FOR FINDING CONVOLUTION IDENTITIES

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1. **Introduction.** The purpose of this paper is to demonstrate a binomial series transformation which generalizes a method used by the author in a previous paper [1] to obtain the well-known series

$$\sum_{k=0}^{\infty} A_k(a, b)z^k = x^a, \tag{1.1}$$

where

$$A_k(a, b) = \frac{a}{a + bk}\binom{a + bk}{k}, \tag{1.2}$$

and $z = (x - 1)/x^b$.

Of course a major application of the expansion (1.1) is to give a simple proof of the generalized Vandermonde convolution identity

$$\sum_{k=0}^{n} (p + qk)A_k(a, b)A_{n-k}(c, b) = \frac{p(a + c) + qan}{a + c} A_n(a + c, b). \tag{1.3}$$

In another paper [2] information may be found concerning a rather similar generalized Abel convolution where coefficients of the form

$$B_k(a, b) = \frac{a}{a + bk}\cdot\frac{(a + bk)^k}{k!} \tag{1.4}$$

occur.

In the present paper we give a number of theorems which arise from a study of the binomial series transformation and which appear to be new results.

2. **The binomial series transformation.**

THEOREM 1. *Let $f(k)$ be independent of n and $f(0) = 1$. Define*

$$F(n) = \sum_{k=0}^{n} (-1)^k\binom{n}{k}\binom{a + bk}{n}f(k). \tag{2.1}$$

Then

$$\sum_{k=0}^{\infty} \binom{a + bk}{k}z^k f(k) = x^a \sum_{n=0}^{\infty} (-1)^n\left(\frac{x - 1}{x}\right)^n F(n), \tag{2.2}$$

where $z = (x - 1)/x^b$.

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