A PROBLEM OF TREYBIG CONCERNING SEPARABLE SPACES

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1. Introduction. In [1], Treybig proved that a metric space X is separable (thus the topology has a countable base) provided X is connected, has no cut point, and has the following property:

For all p, $q \in X$, $p \neq q$, and $\epsilon > 0$, there exists a closed, connected and compact set N such that $N \subset S(p, \epsilon)$ and N separates p from q.

(*Note.* $S(p, \epsilon) = \{x \mid d(p, x) < \epsilon\}$, the open ball with center p and radius ϵ.)

He raised the question as to whether or not the conclusion still holds if the word "compact" in the hypothesis is replaced by the word "separable". The present paper answers this question in the affirmative.

THEOREM. *Suppose X is* (i) *metric,* (ii) *connected,* (iii) *for all $p \in X$, $X \sim \{p\}$ is connected, and* (iv) *for all p, q in X, $p \neq q$, and $\epsilon > 0$, there exists a closed, connected and separable set N such that $N \subset S(p, \epsilon)$ and N separates p from q in X. Then X is separable.*

2. First step and outline of the proof. The result is trivial if X is finite so in the sequel we assume that X is infinite. Apply Hypothesis (iv) to any pair of points p and q, with any $\epsilon > 0$, obtaining a closed, connected and separable point set M_1 separating p from q. Since X is metric and has no cut point, it follows that M_1 is infinite. Let b denote some point of M_1, and let K_1 be an infinite countable set dense in M_1 but not containing b. Later on we will define a separable $M_2 \supset M_1$, and in M_2 pick a countable dense set K_2 not containing b, and in general, for each i, define a separable $M_{i+1} \supset M_i$, and in M_{i+1} a countable dense set K_{i+1} not containing b. We will set $T = (\bigcup_{i=1}^{\infty} K_i)^- = (\bigcup_{i=1}^{\infty} M_i)^-$ and show that $T = X$. Obviously T is separable.

3. Definition of $N(x,n)$, $D(x,n)$, $E(x,n)$, $k(x,n)$. Let x be any point of X except b, and n any positive integer. Apply Hypothesis (iv) with $p = x$, $q = b$. and $\epsilon = 1/n$. Then there exists a closed, connected and separable set N such that

(1) $N \subset S(x, 1/n)$,
(2) $X \sim N = D \cup E$, mutually separated, and
(3) $x \in D$, $b \in E$.

Now D is open, so for some $\alpha > 0$, $S(x, \alpha) \subset D$. Thus there is a smallest positive integer k (greatest $1/k$) such that $S(x, 1/k) \subset D$. The integer k depends upon the particular N selected in $S(x, 1/n)$, and also on the particular way that

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