SOME ORTHOGONAL POLYNOMIALS RELATED TO ELLIPTIC FUNCTIONS

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1. Heine [4, vol. 1; 292–296] has considered the orthogonal polynomials associated with the weight function

$$w(x) = \{x(\alpha - x)(\beta - x)\}^{-\frac{1}{2}} \qquad (0 < x < \alpha; \alpha < \beta).$$

In particular he derives a differential equation of the second order satisfied by these polynomials. Achyeser [1] has investigated the orthogonal polynomials associated with the weight function

$$w(x) = \begin{cases} \{(1 - x^2)(a - x)(b - x)\}^{-\frac{1}{2}} \mid c - x \mid & (-1 \le x \le a,\ b \le x \le 1) \\ 0 & (a < x < b). \end{cases}$$

See also [7; 36]. Each of these sets of polynomials is related to the elliptic functions.

In the present paper we construct sets of orthogonal polynomials suggested by the Stieltjes formulas

$$\int_0^\infty \operatorname{sn}(u, k^2)e^{-zu}\,du = \frac{1}{z^2 + a-}\,\frac{1\cdot 2^2\cdot 3k^2}{z^2 + 3^2a-}\,\frac{3\cdot 4^2\cdot 5k^2}{z^2 + 5^2a-}\,\frac{5\cdot 6^2\cdot 7k^2}{z^2 + 7^2a-}\cdots, \tag{1.1}$$

$$z\int_0^\infty \operatorname{sn}^2(u, k^2)e^{-zu}\,du = \frac{2}{z^2 + 2^2a-}\,\frac{2\cdot 3^2\cdot 4k^2}{z^2 + 4^2a-}\,\frac{4\cdot 5^2\cdot 6k^2}{z^2 + 6^2a-}\,\frac{6\cdot 7^2\cdot 8k^2}{z^2 + 8^2a-}\cdots, \tag{1.2}$$

where in each case $a = k^2 + 1$. These formulas are quoted by Wall [8; 374]; see also Stieltjes [6; 549–554] and Rogers [5]. The polynomials in question are the denominators of the continued fractions occurring in (1.1) and (1.2). In each case we find generating functions for the polynomials and show that they are orthogonal with respect to a discrete weight function. As a matter of fact we do not assume the formulas (1.1) and (1.2) but deal only with the recurrences implied by the continued fractions. (Note that the factor z in the left member of (1.2) is omitted in [6] and [8]).

In the next place (in §§7, 8) we obtain similar results for the continued fractions (also due to Stieltjes) corresponding to the integrals

$$\int_0^\infty \operatorname{cn}(u, k^2)e^{-zu}\,du, \qquad \int_0^\infty \operatorname{dn}(u, k^2)e^{-zu}\,du.$$

There are numerous relations among the polynomials of these four orthogonal systems. Some samples are obtained in §§6 and 9. In particular, certain elliptic function formulas can be utilized to derive various relations among the polynomials; see for example (9.9), (9.10), (9.11) below. Also these poly-

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