APPROXIMATION BY POLYNOMIALS WITH INTEGRAL COEFFICIENTS, A REFORMULATION OF THE STONE-WEIERSTRASS THEOREM

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1. Introduction.

1.1. The Stone-Weierstrass theorem (see for example [9; 9, Theorem 4E]) gives a simple necessary and sufficient condition for polynomials with *real* coefficients to approximate arbitrary continuous functions. In the present paper, we consider the problem of approximating continuous functions by polynomials with *integral* coefficients. Our interest in this problem arose from a discussion with Professor J. R. Isbell. The problem has a long history: J. Pál [11], S. Kakeya [6], Y. Okada [10], S. Bernšteĭn [1], L. Kantorovič [7], and M. Fekete [4] have all studied the problem of approximating continuous functions on intervals on the line by polynomials with integral coefficients. Our results extend those of [11], [6], and [10]. The results of [1], [7], and [4] are related to but not contained in ours. Fekete [2] has also studied approximation by polynomials whose coefficients are certain algebraic integers.

In §§2–5, we study approximation by integral polynomials on the line. With every closed interval on the line of length less than 4, we associate a certain finite subset J. A continuous real function on the interval is arbitrarily uniformly approximable by polynomials with integral coefficients if and only if it is equal to some such polynomial on the set J. For intervals $[\alpha, \beta]$ contained in $[-2, 2]$, the set J is constructed explicitly, and is the set J' described in 5.6. In §6, we extend some of these results to approximation on arbitrary compact Hausdorff spaces. The results in this case are similar to those for intervals on the line, although somewhat more complicated. In special cases, quite explicit results are obtained (see Theorems 6.6 and 6.8).

1.2. Throughout this paper, the symbols α and β will be reserved for real numbers such that $\alpha < \beta$. The symbols $[\alpha, \beta]$ and $]\alpha, \beta[$ designate closed and open intervals, respectively. For a real number a, $[a]$ denotes the integral part of a. The symbol $\mathfrak{C}(\alpha, \beta)$ designates the set of all real-valued continuous functions on $[\alpha, \beta]$.

We make the further conventions:

Q will denote a polynomial in the real variable x with integral coefficients (in §§1–5, not in §6);

R will denote a polynomial in the real variable x with integral coefficients and leading coefficient 1;

"f is approximable on $[\alpha, \beta]$" will mean that $f \in \mathfrak{C}(\alpha, \beta)$ and for every positive

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