AN ANALOGUE OF MERCER'S THEOREM

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1. **Introduction.** In recent years Mautner [3], Gårding [2], and Bade and Schwartz [1], have obtained sufficient conditions that a self-adjoint operator have a generalized eigenfunction expansion. It is clear that by using such expansions one should be able to prove an expansion theorem for a class of kernels that would be analogous to Mercer's theorem for the class of kernels associated with completely continuous operators. Such a theorem is proved in §3. In §2 we give the preliminary material that is needed in §3.

Our starting point will be Gårding's theorem which is stated in §2. In his paper [2] Gårding uses a direct integral decomposition of the underlying Hilbert space while we prefer to use the decomposition (explained in §2) that is used in [1] and [3]. The job of translating Gårding's work to the decomposition used here is completely straightforward and will be omitted.

2. **Gårding's theorem.** Let X be a closed subset of Euclidean N-space and m a Radon measure on X. Let $L_2(m)$ be the usual complex Hilbert space over (X, m). It is well known that $L_2(m)$ is separable. Let T be a (possibly unbounded) self-adjoint operator in $L_2(m)$, then if $E(\lambda)$ is the spectral resolution of T, a version of the spectral theorem asserts that there exists a sequence of elements, $\{f_n\}$, in $L_2(m)$ such that $L_2(m)$ is the orthogonal direct sum $\sum_1^\infty \mathcal{H}_n$, where $\mathcal{H}_n$ is the closed manifold of all elements of the form $F(T)f_n$ where $F \in L_2(\mu_n)$ and $\mu_n(\cdot) = (E(\cdot)f_n, f_n)$. In fact the correspondence $F(T)f_n \to F(\cdot)$ establishes an isomorphism, U_n, of $\mathcal{H}_n$ onto $L_2(\mu_n)$. Moreover the relationship

$$(Uf)_n = U_n(f)_n \tag{2.1}$$

establishes an isomorphism U of $L_2(m)$ onto the direct sum $\mathcal{S} = \sum_1^\infty L_2(\mu_n)$ which diagonalizes T in the sense that

$$(U\Phi(T)f)_n(\lambda) = \Phi(\lambda)(Uf)_n(\lambda) \tag{2.2}$$

for all $f \in \mathcal{D}(\Phi(T))$. (If $f \in L_2(m)$ then $(f)_n$ denotes the component of f in $\mathcal{H}_n$ and $(Uf)_n = U_n(f)_n$ is the component of Uf in $L_2(\mu_n)$.) Also each $\mathcal{H}_n$ reduces T and the support of each μ_n is contained in the spectrum of T. Finally the f_n and hence the μ_n may be chosen so that $\mu_1 \gg \mu_2 \gg \cdots$, where $\mu \gg \nu$ means ν is absolutely continuous with respect to μ. With this choice the μ_n are unique up to equivalence of measures. We adopt the convention that all summations are with respect to the index n which runs from 1 to ∞ unless explicitly stated otherwise.

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