LIMITS FOR THE CHARACTERISTIC ROOTS OF A MATRIX. VII

By Alfred Brauer

In the recently published interesting book of E. Bodewig, *Matrix Calculus* [1; 67] some results of the earlier parts of my paper [2] are mentioned. It is stated that I obtain $n(n-1)/2$ ovals of Cassini which contain the characteristic roots of a given square matrix of order n instead of the well-known n circles for these roots. But Bodewig adds that "in my opinion this result is not a practical improvement. For firstly the ovals are more difficult to draw and secondly if $n = 10$ or 20 or more, there are far too many to permit any practical computation. Yet in theoretical importance the new theorems are interesting."

It seems therefore necessary to discuss my results from a practical standpoint.

We use in the following the circles and ovals obtained from the rows. In the same way we could use the columns.

Let $A = (a_{\kappa\lambda})$ be a square matrix of order n. We set

$$\sum_{\substack{\nu=1 \\ \nu\neq\kappa}}^{n} | a_{\kappa\nu} | = P_\kappa \qquad (\kappa = 1, 2, \cdots, n).$$

Then each characteristic root lies in at least one of the n circles

$$(155) \qquad | z - a_{\kappa\kappa} | \leq P_\kappa \qquad (\kappa = 1, 2, \cdots, n),$$

and the theorem on the ovals in its simplest form [2, II, Theorem 11] states that each root lies in at least one of the $n(n-1)/2$ ovals

$$(156) \qquad | z - a_{\kappa\kappa} | \, | z - a_{\lambda\lambda} | \leq P_\kappa P_\lambda \qquad (\kappa, \lambda = 1, 2, \cdots, n; \kappa \neq \lambda).$$

It must be remarked that for the applications the ovals should not replace the circles. If some information is needed on the location of the characteristic roots of a given matrix, then first we shall use the circles (155). If the results obtained in this way are not sufficient, then we shall try to use the ovals.

If, for instance, we already know that a root lies in the first of the circles (155), then we do not have to consider all the $n(n-1)/2$ ovals in order to obtain better bounds for this root, but only the $n-1$ ovals

$$| z - a_{11} | \, | z - a_{\nu\nu} | \leq P_1 P_\nu \qquad (\nu = 2, 3, \cdots, n).$$

Often only a few of these ovals actually have to be considered.

The problem we have to discuss is the following one. Is it easily possible to replace the union G of the n circles (155) by a smaller region G' by using the ovals (156)?

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